Existence of Periodic Orbits in Completed Lagrangian Hybrid Systems with Non-Plastic Collisions

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Abstract. In this paper, we consider hybrid models of mechanical systems undergoing impacts, i.e., Lagrangian hybrid systems, and study their periodic orbits in the presence of Zeno behavior. The main result of this paper is explicit conditions under which the existence of stable periodic orbits for a Lagrangian hybrid system with *plastic impacts* implies the existence of periodic orbits in the same Lagrangian hybrid systems with *non-plastic impacts*. Since non-plastic impacts result in Zeno behavior, in proving this result we necessarily obtain an understanding of periodic orbits containing Zeno behavior. These results are practically useful to a wide range of mechanical systems, as demonstrated through the example of a double pendulum with a mechanical stop.

1 Introduction

Periodic orbits play a fundamental role in the design and analysis of hybrid systems modeling a myriad of applications ranging from biological systems to chemical processes to robotics. To provide a concrete example, bipedal robots are naturally modeled by hybrid systems [9, 15]. The entire process of obtaining walking gaits for bipedal robots can be viewed simply as designing control laws that create stable periodic orbits in a specific hybrid system. This is a theme that is repeated throughout the various applications of hybrid systems.

In order to better understand the role that periodic orbits play in hybrid systems, we must first restrict our attention to hybrid systems that model a wide range of physical systems but are simple enough to be amenable to analysis. In this light, we consider *Lagrangian hybrid systems* modeling mechanical systems undergoing impacts; systems of this form have a rich history and are useful in a wide-variety of applications [5]. In particular, a *hybrid Lagrangian* consists of a configuration space, a Lagrangian modeling a mechanical systems, and a *unilateral constraint function* that gives the set of admissible configurations for this system. From this data, we obtain a Lagrangian hybrid system. The benefit of studying systems of this form is that they often display Zeno behavior (when an infinite number of collisions occur in a finite amount of time), so they give an ideal class of system in which to gain an intuitive understanding of Zeno behavior and its relationship to periodic orbits in hybrid systems which is the main focus of this paper.

Before discussing the type of periodic orbits that will be studied in this paper, we must first explain how one deals with Zeno behavior in Lagrangian hybrid systems by *completing* the hybrid model of these systems. Using the special structure of Lagrangian hybrid systems, the main observation is that points to which Zeno executions converge—*Zeno points*—must satisfy constraints imposed by the unilateral constraint function. These constraints are *holonomic* in nature, which implies that after the Zeno point the hybrid system should switch to a holonomically constrained dynamical system evolving on the zero level set of the unilateral constraint function. Moreover, if the force constraining the dynamical system to the surface becomes zero, there should be a switch back to the original hybrid system. These observations allow one to formally complete a Lagrangian hybrid system by adding an additional *post-Zeno* domain to the system [2, 20].

In this paper, we study periodic orbits for completed Lagrangian hybrid systems that pass through both the original and the *post-Zeno* domain of the hybrid system; while periodic orbits of this form have never been studied before they are of paramount importance to a wide variety of applications, e.g., this is the type of orbit one obtains in bipedal robots. In particular, we begin by considering a *simple* periodic orbit which is an orbit for a Lagrangian hybrid system with *perfectly plastic impacts*, i.e., at a collision event, the system instantly switches to the post-Zeno domain. The question is: *what happens to a simple periodic orbit when the impacts are not perfectly plastic*? The main result of this paper guarantees existence of a periodic orbit for completed Lagrangian hybrid system with non-plastic impacts given a stable periodic for the same system with plastic impacts; moreover, we give explicit bounds on the degree of plasticity that ensures the existence of such orbit.

The importance of the main result of this paper lies in the fact that impacts in mechanical systems are never perfectly plastic, so it is important to understand what happens to periodic orbits for perfectly plastic impacts in the case of non-plasticity. Using the example of a bipedal robot with knees [9, 15], the knee locking (leg straightening) is modeled as a perfectly plastic impact. If one were to find a walking gait for this biped under this assumption, the main result of this paper would ensure that there would also be a walking gait in the case when the knee locking is not perfectly plastic as would be true in reality. In light of this example, we conclude the paper by applying the main result of this paper to a double pendulum with a mechanical stop, which models a single leg of a bipedal robot with knees.

Both periodic orbits and Zeno behavior have been well-studied in the literature although they have yet to be studied simultaneously. An exception is the work in [4], which focuses on design of stable tracking control for cyclic tasks with Zeno behavior in Lagrangian hybrid systems, in case where the system is *fully* *actuated*. Note, however, that these techniques cannot, in general, be applied to locomotion systems, which are typically *underactuated*, i.e. have uncontrolled degrees-of-freedom. With regard to Zeno behavior, it has been studied in the context of mechanical systems in [13, 16], and [19] with results that complement the results of this paper, and studied for other hybrid models in [6, 11, 22, 23]. Periodic orbits have primarily been studied in hybrid systems in the context of bipedal walking in [9, 10, 17] and running [8].

2 Lagrangian Hybrid Systems

In this section, we introduce the notion of a hybrid Lagrangian, the associated Lagrangian hybrid system, and discuss Zeno behavior in systems of this form. Hybrid Lagrangians of this form have been studied in the context of Zeno behavior and reduction; see [1], [13], and [19]. We begin this section by reviewing the notion of a simple hybrid system.

Definition 1. *A simple hybrid system is a tuple:*

$$\mathscr{H} = (D, G, R, f),$$

where

- D *is a smooth manifold called the domain,*
- G *is an embedded submanifold of* D *called the guard,*
- R *is a smooth map* $R : G \to D$ *called the reset map,*
- f *is a vector field on the manifold* D*.*

Hybrid executions. A *hybrid execution* of a simple hybrid system $\mathscr{H}$ is a tuple $\chi = (\Lambda, \mathcal{I}, \mathcal{C})$, where

- $\Lambda = \{0, 1, 2, \ldots\} \subseteq \mathbb{N}$ is an indexing set.
- $\mathcal{I} = \{I_i\}_{i\in\Lambda}$ is a *hybrid interval* where $I_i = [t_i, t_{i+1}]$ if $i, i+1 \in \Lambda$ and $I_{N-1} = [t_{N-1}, t_N]$ or $[t_{N-1}, t_N)$ or $[t_{N-1}, \infty)$ if $|\Lambda| = N$, N finite. Here, $t_i, t_{i+1}, t_N \in \mathbb{R}$ and $t_i \leq t_{i+1}$.
- $\mathcal{C} = \{c_i\}_{i\in\Lambda}$ is a collection of integral curves of f, i.e., $\dot{c}_i(t) = f(c_i(t))$ for $t \in I_i$, $i \in \Lambda$,

And the following conditions hold for every $i, i+1 \in \Lambda$:

$$\begin{array}{ll} \text{(i)} & c_i(t_{i+1}) \in G, \\ \text{(ii)} & R(c_i(t_{i+1})) = c_{i+1}(t_{i+1}), \\ \text{(iii)} & t_{i+1} = \min\{t \in I_i : c_i(t) \in G\}. \end{array}$$

The *initial condition* for the hybrid execution is $c_0(t_0)$.

Lagrangians. Let $q \in I\!R^n$ be the *configuration* of a mechanical system[3]. In this paper, we will consider Lagrangians, $L : I\!R^{2n} \to I\!R$, describing mechanical, or robotic, systems, which are Lagrangians of the form

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q), \tag{1}$$

where $M(q)$ is the (positive definite) inertial matrix, $\frac{1}{2}\dot{q}^T M(q)\dot{q}$ is the kinetic energy and $V(q)$ is the potential energy. We will also consider a *control law* $u(q, \dot{q})$, which is a smooth function $u : I\!R^{2n} \to I\!R^n$. In this case, the Euler-Lagrange equations yield the (unconstrained, controlled) equations of motion for the system:

$$M(q)\ddot{q} + C(q, \dot{q}) + N(q) = u(q, \dot{q}), \tag{2}$$

where $C(q, \dot{q})$ is the vector of centripetal and Coriolis terms (cf. [18]) and $N(q) = \frac{\partial V}{\partial q}(q)$. Setting $x = (q, \dot{q})$, the Lagrangian vector field, f_L, associated to L takes the familiar form:

$$\dot{x} = f_L(x) = \begin{pmatrix} \dot{q} \\ M(q)^{-1}(-C(q, \dot{q}) - N(q) + u(q, \dot{q})) \end{pmatrix}. \tag{3}$$

This process of associating a dynamical system to a Lagrangian will be mirrored in the setting of hybrid systems. First, we introduce the notion of a hybrid Lagrangian.

Definition 2. *A simple hybrid Lagrangian is defined to be a tuple*

$$\mathbf{L} = (Q, L, h),$$

where

- Q *is the configuration space,*
- $L : TQ \to I\!R$ *is a hyperregular Lagrangian,*
- $h : Q \to I\!R$ *provides a unilateral constraint on the configuration space; we assume that* $h^{-1}(0)$ *is a smooth manifold.*

Simple Lagrangian hybrid systems. For a Lagrangian (1), there is an associated dynamical system (3). Similarly, given a hybrid Lagrangian $\mathbf{L} = (Q, L, h)$ the *simple Lagrangian hybrid system* associated to $\mathbf{L}$ is the simple hybrid system:

$$\mathscr{H}_{\mathbf{L}} = (D_{\mathbf{L}}, G_{\mathbf{L}}, R_{\mathbf{L}}, f_{\mathbf{L}}).$$

First, we define

$$\begin{aligned} D_{\mathbf{L}} &= \{(q, \dot{q}) \in TQ : h(q) \geq 0\}, \\ G_{\mathbf{L}} &= \{(q, \dot{q}) \in TQ : h(q) = 0 \text{ and } dh(q)\dot{q} \leq 0\}, \end{aligned}$$

[3] For simplicity, we assume that the configuration space is identical to $I\!R^n$

where

$$dh(q) = \left(\frac{\partial h}{\partial q}(q)\right)^T = \left(\frac{\partial h}{\partial q_1}(q) \cdots \frac{\partial h}{\partial q_n}(q)\right).$$

In this paper, we adopt the reset map ([5]):

$$R_{\mathbf{L}}(q,\dot{q}) = (q, P_{\mathbf{L}}(q,\dot{q})),$$

which based on the *impact equation*

$$P_{\mathbf{L}}(q,\dot{q}) = \dot{q} - (1+e)\frac{dh(q)\dot{q}}{dh(q)M(q)^{-1}dh(q)^T}M(q)^{-1}dh(q)^T, \tag{4}$$

where $0 \leq e \leq 1$ is the *coefficient of restitution*, which is a measure of the energy dissipated through impact. This reset map corresponds to rigid-body collision law under the assumption of *frictionless impact*. Examples of more complicated collision laws that account for friction can be found in [5] and [7]. Finally, $f_{\mathbf{L}} = f_L$ is the Lagrangian vector field associated to L in (3).

3 Completed Hybrid Systems

In this section we introduce the notion of a completed hybrid system [2], [20], and define the notions of simple periodic orbit and Zeno periodic orbit, corresponding to periodic completed executions under plastic and non-plastic collisions. Then we define the stability of periodic orbits.

Choice of coordinates. In the rest of this paper, we assume that the generalized coordinates contain the constraint function h as a coordinate, i.e. $q = (z, h)$. This assumption is quite general, since a transformation to such coordinate set must exist, at least locally, due to the regularity of $h(q)$. The state of the system thus takes the form $x = (z, h, \dot{z}, \dot{h}) \in I\!R^{2n}$. When the coordinates take this special form, the reset map (4) simplifies to

$$P_{\mathbf{L}}(q,\dot{q}) = \begin{pmatrix} \dot{z} - (1+e)\dot{h}\eta(z) \\ -e\dot{h} \end{pmatrix}, \qquad \text{where} \qquad \eta(z) = \left.\frac{[M^{-1}(q)]_{1\ldots n-1}, n}{[M^{-1}(q)]_{n,n}}\right|_{h=0}. \tag{5}$$

The instantaneous solution for the accelerations $\ddot{q}$ in (2) is given by

$$\ddot{q}(q,\dot{q}) = (\ddot{z}(q,\dot{q}), \ddot{h}(q,\dot{q})) = M(q)^{-1}\left(u(q,\dot{q}) - C(q,\dot{q}) - N(q)\right). \tag{6}$$

Zeno behavior. A hybrid execution χ is *Zeno* if $\Lambda = \mathbb{N}$ and

$$\lim_{i\to\infty} t_i = t_\infty < \infty.$$

Here t_∞ is called the *Zeno time*. If χ is a Zeno execution of a Lagrangian hybrid system $\mathscr{H}_{\mathbf{L}}$, then its *Zeno point* is defined to be

$$x_\infty = (q_\infty, \dot{q}_\infty) = \lim_{i\to\infty} c_i(t_i) = \lim_{i\to\infty}(q_i(t_i), \dot{q}_i(t_i)).$$

These limit points necessarily lie on the *constraint surface* in state space, defined by:

$$\mathcal{S} = \{(q,\dot{q}) \in I\!R^{2n} : h = \dot{h} = 0\}.$$

Constrained dynamical systems. We now define the holonomically constrained dynamical system $\mathscr{D}_{\mathbf{L}}$ associated with the hybrid Lagrangian $\mathbf{L}$. For such systems, the constrained equations of motion can be obtained from the equations of motion for the unconstrained system (2), and are given by (cf. [18])

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = dh(q)^T\lambda + u(q,\dot{q}), \tag{7}$$

where λ is the Lagrange multiplier which represents the contact force. Differentiating the constraint equation $h(q) = 0$ twice with respect to time and substituting the solution for $\ddot{q}$ in (7), the solution for the constraint force is obtained as follows:

$$\begin{aligned}\lambda(q,\dot{q}) = &\left(dh(q)M(q)^{-1}dh(q)^T\right)^{-1} \\ &\left(dh(q)M(q)^{-1}(C(q,\dot{q})\dot{q} + N(q) - u(q,\dot{q})) - \dot{q}^T H(q)\dot{q}\right).\end{aligned} \tag{8}$$

When the coordinates are of the form $q = (z, h)$, the solution for λ in (8) is evaluated at $h = \dot{h} = 0$, and can be simplified to the form

$$\lambda(z,\dot{z}) = \eta(z)^T(C(z,\dot{z}) + N(z) - u(z,\dot{z})).$$

From the constrained equations of motion, for $x = (q,\dot{q})$, we get the vector field

$$\dot{x} = \tilde{f}_L(x) = \begin{pmatrix} \dot{q} \\ M(q)^{-1}(-C(q,\dot{q})\dot{q} - N(q) + u(q,\dot{q}) + dh(q)^T\lambda(q,\dot{q})) \end{pmatrix}$$

Note that $\tilde{f}_L$ defines a vector field on the manifold $TQ|_{h^{-1}(0)}$, from which we obtain the dynamical system $\mathscr{D}_{\mathbf{L}} = (TQ|_{h^{-1}(0)}, \tilde{f}_L)$. For this dynamical system, $q(t)$ slides along the surface $h^{-1}(0)$ as long as the constraint force λ is positive.

A *constrained execution* $\tilde{\chi}$ of $\mathscr{D}_{\mathbf{L}}$ is a pair $(\tilde{I}, \tilde{c})$ where $\tilde{I} = [\tilde{t}_0, \tilde{t}_f] \subset I\!R$ if $\tilde{t}_f$ is finite and $\tilde{I} = [\tilde{t}_0, \tilde{t}_f) \subset I\!R$ if $\tilde{t}_f = \infty$ and $\tilde{c} : \tilde{I} \to TQ$, with $\tilde{c}(t) = (q(t), \dot{q}(t))$ a solution to the dynamical system $\mathscr{D}_{\mathbf{L}}$ satisfying the following properties:

$$\begin{array}{ll} \text{(i)} & h(\tilde{t}_0) = 0, \\ \text{(ii)} & \dot{h}(\tilde{t}_0) = 0, \\ \text{(iii)} & \lambda(q(\tilde{t}_0), \dot{q}(\tilde{t}_0)) > 0, \\ \text{(iv)} & \tilde{t}_f = \min\{t \in \tilde{I} : \lambda(q(t), \dot{q}(t)) = 0\}. \end{array} \tag{9}$$

Using the notation and concepts introduced thus far, we introduce the notion of a completed hybrid system.

Definition 3. *If* $\mathbf{L}$ *is a simple hybrid Lagrangian and* $\mathscr{H}_{\mathbf{L}}$ *the corresponding Lagrangian hybrid system, the corresponding completed Lagrangian hybrid system*[4] *is defined to be:*

$$\overline{\mathscr{H}}_{\mathbf{L}} := \begin{cases} \mathscr{D}_{\mathbf{L}} & \text{if} \quad h = 0\,,\ \dot{h} = 0, \text{ and } \lambda(q,\dot{q}) > 0 \\ \mathscr{H}_{\mathbf{L}} & \qquad\qquad \text{otherwise.} \end{cases}$$

[4] As was orginally pointed out in [2], this terminology (and notation) is borrowed from topology, where a metric space can be completed to ensure that "limits exist."

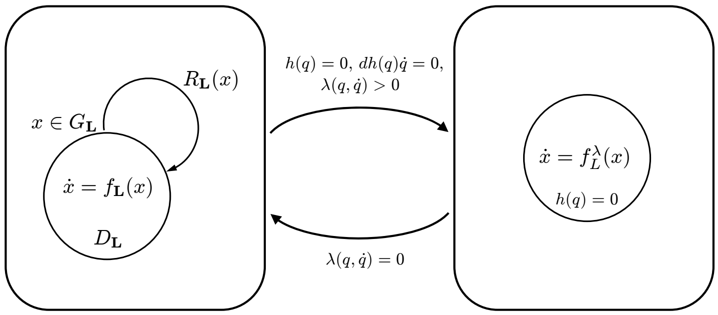

Fig. 1. A graphical representation of a completed hybrid system.

Remarks. The system $\overline{\mathscr{H}}_{\mathbf{L}}$ can be viewed simply as a hybrid system; in this case, the reset maps are the identity, and the guards are given as in Fig. 1. Also note that the only way for the transition to be made from the hybrid system $\mathscr{H}_{\mathbf{L}}$ to the constrained system $\mathscr{D}_{\mathbf{L}}$ is if a specific Zeno execution reaches its Zeno point. Second, a transition for $\mathscr{D}_{\mathbf{L}}$ to $\mathscr{H}_{\mathbf{L}}$ happens when the constraint force λ crosses zero. Finally, note that the definitions of $\ddot{h}(q,\dot{q})$ in (6) and $\lambda(q,\dot{q})$ in (8) imply that while sliding along the surface $h^{-1}(0)$, either $\ddot{h}=0$ and $\lambda>0$, corresponding to maintaining constrained motion, or $\ddot{h}>0$ and $\lambda=0$, corresponding to leaving the constraint surface and switching back to the hybrid system. Thus, the definition of the completed hybrid system is consistent.

The completed execution. Having introduced the notion of a completed hybrid system, we must introduce the semantics of solutions of systems of this form. That is, we must introduce the notion of a *completed execution* of a completed hybrid system.

Definition 4. *Given a simple hybrid Lagrangian* $\mathbf{L}$ *and the associated completed system* $\overline{\mathscr{H}}_{\mathbf{L}}$*, a* completed execution $\overline{\chi}$ *is a (possibly infinite) ordered sequence of alternating constrained and hybrid executions*

$$\overline{\chi}=\{\tilde{\chi}^{(1)},\ \chi^{(2)},\ \tilde{\chi}^{(3)},\ \chi^{(4)},\ \ldots\},$$

with $\tilde{\chi}^{(i)}$ *and* $\chi^{(j)}$ *executions of* $\mathscr{D}_{\mathbf{L}}$ *and* $\mathscr{H}_{\mathbf{L}}$*, respectively, that satisfy the following conditions:*

(i) For each pair $\tilde{\chi}^{(i)}$ *and* $\chi^{(i+1)}$*,*
$\tilde{t}_f^{(i)}=t_0^{(i+1)}$ *and* $\tilde{c}^{(i)}(\tilde{t}_f^{(i)})=c_0^{(i+1)}(t_0^{(i+1)})$.

(ii) For each pair $\chi^{(i)}$ *and* $\tilde{\chi}^{(i+1)}$*,*
$t_\infty^{(i)}=\tilde{t}_0^{(i+1)}$ *and* $c_\infty^{(i)}=\tilde{c}^{(i+1)}(\tilde{t}_0^{(i+1)})$.

where the superscript (i) *denotes values corresponding to the* i^{th} *execution in* $\overline{\chi}$*, and* $t_\infty^{(i)}, c_\infty^{(i)}$ *denote the Zeno time and Zeno point associated with the* i^{th} *hybrid execution* $\chi^{(i)}$*.*

Periodic orbits of completed hybrid systems. In the special case of plastic collisions $e = 0$, a *simple periodic orbit* is a completed execution $\overline{\chi}$ with initial conditions $\tilde{c}^{(1)}(0) = x^*$ that satisfies $\tilde{c}^{(3)}(\tilde{t}_0^{(3)}) = x^*$. The period of $\overline{\chi}$ is $T = \tilde{t}_0^{(3)}$. In other words, this orbit consists of a constrained execution starting at x^*, followed by a hybrid (unconstrained) execution which is ended by a single plastic collision at $t = T$, that resets the state back to x^*.

For non-plastic collisions $e \neq 0$, a *Zeno periodic orbit* is a completed execution $\overline{\chi}$ with initial conditions $\tilde{c}^{(1)}(0) = x^*$ that satisfies $c_\infty^{(2)} = \tilde{c}^{(3)}(\tilde{t}_0^{(3)}) = x^*$. The period of $\overline{\chi}$ is $T = t_\infty^{(2)} = \tilde{t}_0^{(3)}$. In other words, this orbit consists of a constrained execution starting at x^*, followed by a Zeno execution which converges in finite time back to x^*.

Stability of hybrid periodic orbits. We now define the stability of hybrid periodic orbits.

Definition 5. *A Zeno (or simple) periodic orbit* $\overline{\chi} = \{\tilde{\chi}^{(1)},\ \chi^{(2)},\ \tilde{\chi}^{(3)},\ \chi^{(4)},\ ...\}$ *with initial conditions* $x^* \in \mathcal{S}$ *is* locally exponentially stable *if there exist a neighborhood* $U \subset \mathcal{S}$ *of* x^* *and a scalar* $\gamma \in (0, 1)$ *such that for any initial conditions* $x_0 = \tilde{c}^{(1)}(0) \in U$*, the resulting completed execution satisfies* $\|\tilde{c}^{(2k+1)}(\tilde{t}_0^{(2k+1)}) - x^*\| \leq \|x_0 - x^*\|\gamma^k$ *for* $k = 1, 2, \ldots$*.*

4 Main Result

In this section we present the main result of this paper, namely, conditions under which the existence and stability of a simple periodic orbit imply existence of a Zeno periodic orbit.

4.1 Statement of Main Result

Before stating this result, some preliminary setup is needed.

We can write $x^* = (z^*, 0, \dot{z}^*, 0)$, and define three types of neighborhoods of x^* in three different subspaces of $\mathbb{R}^{2n}$. For $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 > 0$, the neighborhoods $\Omega_1(\epsilon_1)$, $\Omega_2(\epsilon_1, \epsilon_2)$ and $\Omega_4(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ are defined as follows.

$$\begin{aligned}
\Omega_1(\epsilon_1) &= \{(q, \dot{q}) : \quad h = 0,\ \dot{h} = 0,\ \text{and } \|z - z^*\| < \epsilon_1\} \\
\Omega_2(\epsilon_1, \epsilon_2) &= \{(q, \dot{q}) : \quad h = 0,\ \dot{h} = 0,\ \|z - z^*\| < \epsilon_1, \text{ and } \|\dot{z} - \dot{z}^*\| < \epsilon_2\} \\
\Omega_4(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) &= \{(q, \dot{q}) : \quad \|z - z^*\| < \epsilon_1,\ \|\dot{z} - \dot{z}^*\| < \epsilon_2, \\
&\qquad\qquad 0 < h < \epsilon_3, \text{ and } |\dot{h}| < \epsilon_4\Big\}
\end{aligned} \tag{10}$$

Assume we are given a control law $u(q, \dot{q})$ and a starting point $x^* \in \mathcal{S}$ for which there exists of a simple periodic periodic orbit $\overline{\chi}^*$ starting at x^* which is locally exponentially stable. Let $v^* = \left|\dot{h}_0^{(2)}(t_1^{(2)})\right|$, which is the pre-collision velocity at the single (plastic) collision in the periodic orbit. The following assumption is an implication of the stability of $\overline{\chi}^*$:

Assumption 1. *Assume that there exist $\epsilon_1, \epsilon_2 > 0$,$\kappa \geq 1$ and $\gamma \in (0,1)$, such that for any initial conditions $x_0 \in \Omega_2(\epsilon_1, \epsilon_2)$, the corresponding completed execution with $e = 0$ satisfies the two following requirements are satisfied:*

$$\begin{array}{ll} (a) & \tilde{c}^{(3)}(\tilde{t}_0^{(3)}) \in \Omega_2(\gamma\epsilon_1, \gamma\epsilon_2) \\ (b) & \left|\dot{h}_0(t_1^{(2)})\right| < \kappa v^*. \end{array} \tag{11}$$

Setup. To provide the conditions needed for the main result, for the given ϵ_1, ϵ_2 and κ, let the neighborhood Ω be defined as

$$\Omega = \Omega_4(\epsilon_1, \epsilon_2, \epsilon_3, \kappa v^*) \tag{12}$$

for some $\epsilon_3 > 0$, and define the following scalars:

$$\begin{aligned} a_{min} &= -\max\nolimits_{(q,\dot{q})\in\Omega} \ddot{h}(q,\dot{q}) \\ a_{max} &= -\min\nolimits_{(q,\dot{q})\in\Omega} \ddot{h}(q,\dot{q}) \\ \delta &= \sqrt{\left|\tfrac{a_{max}}{a_{min}}\right|} \\ \dot{z}_{max} &= \|\dot{z}^*\| + \epsilon_2 \\ \ddot{z}_{max} &= \max\nolimits_{(q,\dot{q})\in\Omega} \|\ddot{z}(q,\dot{q})\| \\ \eta_{max} &= \max\nolimits_{z\in\Omega_1(\rho_1)} \|\eta(z)\|. \end{aligned} \tag{13}$$

The following theorem states the sufficient conditions for existence of a Zeno periodic orbit given a simple periodic orbit.

Theorem 1. *Consider a simple periodic orbit $\overline{\chi}^*$ which is locally exponentially stable, and the given $\epsilon_1, \epsilon_2 > 0$, $\kappa \geq 1$, and $\gamma \in (0,1)$ that satisfy Assumption 1. Then for a given coefficient of restitution e, if the neighborhood Ω defined in (12), and the associated scalars defined in (13) satisfy the following conditions:*

$$a_{max} \geq a_{min} > 0 \tag{14}$$

$$e\delta^2 < 1 \tag{15}$$

$$\frac{2e\kappa v^*}{a_{min}(1-\delta^2 e)}\dot{z}_{max} \leq \epsilon_1(1-\gamma) \tag{16}$$

$$\left(\frac{1+\delta}{1-\delta e}\eta_{max} + \frac{2}{a_{min}(1-\delta^2 e)}\ddot{z}_{max}\right) e\kappa v^* \leq \epsilon_2(1-\gamma), \tag{17}$$

$$\frac{(e\kappa v^*)^2}{2a_{min}} \leq \epsilon_3 \tag{18}$$

then there exists a Zeno periodic orbit with initial conditions within $\Omega_2(\epsilon_1, \epsilon_2)$.

4.2 Proof of the Main Result

Before proving Theorem 1, we must define some preliminary notation. Consider the completed execution $\overline{\chi}$ with $e = 0$, and the execution $\overline{\chi}'$ with $e > 0$ under the same given initial conditions $x_0 \in \Omega_2(\epsilon_1, \epsilon_2)$. Since we are only interested in the first hybrid and constrained elements of $\overline{\chi}$ and $\overline{\chi}'$, respectively, we simplify the notation by defining $\overline{\chi} = \{\tilde{\chi},\ \chi, ...\}$ and $\overline{\chi}' = \{\tilde{\chi}',\ \chi', ...\}$. Since the constrained motion does not contain any collisions, it is clear that $\tilde{\chi} = \tilde{\chi}'$. Moreover, the hybrid executions χ and χ are also identical until the first collision time, that is $c_0(t) = c_0'(t)$ for $t \in [t_0, t_1]$ and $t_1 = t_1'$. Therefore, we will compare the solutions $c_i'(t)$ and $c_i(t)$ for $i > 1$, i.e. after the time t_1.

We now give the outline of the proof, which is divided into three steps. The first step proves that if the hybrid execution χ' stays within the neighborhood Ω, then conditions (14) and (15) imply that it is a Zeno execution. Step 2 verifies that under conditions (16) and (17), the execution χ' actually stays within Ω. The final step then shows that conditions (16) and (17) also imply the existence of a Zeno periodic orbit, whose starting point at lies within $\Omega_2(\epsilon_1, \epsilon_2)$. The results of these two steps are stated as two lemmas. Finally, the third step utilizes the previous steps to complete the proof of Theorem 1.

Step 1. Consider a neighborhood Ω that satisfies conditions (14) and (15). We now assume that the trajectory of the hybrid execution satisfies $c_i'(t) \in \Omega$ for all $t \in I_i',\ i \in \Lambda' - \{0\}$. This assumption implies that the h-component of the execution $c_i'(t) = (z_i'(t), h_i'(t), \dot{z}_i'(t), \dot{h}_i'(t))$ satisfies the second-order differential inclusion

$$\ddot{h}_i'(t) \in [-a_{max}, -a_{min}], \tag{19}$$

for all $t \in I_i',\ i \in \Lambda' - \{0\}$. At each collision time $t_i',\ i \in \Lambda' - \{0\}$, (19) is re-initialized according to the collision law (5) as

$$\dot{h}_{i+1}'(t_i') = -e\dot{h}_i'(t_i'), \quad h_{i+1}'(t_i') = h_i'(t_i') = 0. \tag{20}$$

Let $\tau_i = t_{i+1}' - t_i'$, which is the time difference between consecutive collisions, and let $v_i = -\dot{h}_{i-1}'(t_i')$, which is the pre-collision velocity at time t_i'. The following lemma summarizes results on the execution of the differential inclusion (19) with the re-initialization rule (20).

Lemma 1. *Assume that the hybrid execution χ' satisfies the differential inclusion (19) for all $t \in I_i',\ i \in \Lambda' - \{0\}$, and that a_{min}, a_{max}, and δ satisfy the conditions (14) and (15). Then χ' is a Zeno execution with a Zeno time t_∞.*

Moreover, the solution $c_i'(t)$ satisfies the following for all $i \geq 1$

$$v_i \leq v_1 (e\delta)^{i-1} \tag{21}$$

$$\left|\dot{h}_i'(t)\right| \leq v_1 \text{ for all } t \in I_i' \tag{22}$$

$$\tau_i \leq \frac{2ev_1}{a_{min}} (e\delta^2)^{i-1} \tag{23}$$

$$t_\infty' - t_1' \leq \frac{2ev_1}{a_{min}(1 - e\delta^2)} \tag{24}$$

$$h_i'(t) \leq \frac{e^2 v_1^2}{2a_{min}} \text{ for all } t \in I_i'. \tag{25}$$

The proof of Lemma 1, which utilizes methods of optimal control, appears in Appendix A.

Step 2: We now verify that for initial conditions in $\Omega_2(\epsilon_1, \epsilon_2)$, the solution actually stays within Ω, as summarized in the following lemma.

Lemma 2. *Consider a neighborhood Ω that satisfies conditions (14)-(17). Then for any initial conditions $x_0 \in \Omega_2(\epsilon_1, \epsilon_2)$, the hybrid execution χ' is a Zeno execution that satisfies $c_i'(t) \in \Omega$ for all $t \in I_i'$, $i \in \Lambda' - \{0\}$.*

Proof. Let us denote $x_1 = c_1(t_1) = (z_1, 0, \dot{z}_1, 0)$, which is the post-collision state at time $t_1 = t_1'$ *under a plastic collision*. We now verify that under the bounds implied by the neighborhood Ω, $c_i'(t)$ is guaranteed to stay within Ω for all $t \in I_i'$, $i \geq 1$. First, consider the z-component of $c_i'(t)$. Since $z_i'(t_i') = z_{i+1}'(t_i')$, i.e. z does not change at collision times, it changes only during the continuous phases. Therefore, it satisfies:

$$z_i'(t) \leq \dot{z}_{max}(t - t_1) \leq \dot{z}_{max}(t_\infty' - t_1') \leq \frac{2\kappa v^*}{a_{min}(1 - \delta^2 e)}.$$

According to Assumption 1, $\|z_1 - z^*\| < \gamma\epsilon_1$. Therefore, condition (16), along with the triangle inequality imply that

$$\|z_i'(t) - z^*\| \leq \|z_i'(t) - z_1\| + \|z_1 - z^*\| \leq (1 - \gamma)\epsilon_1 + \gamma\epsilon_1 = \epsilon_1,$$

and the z-component of $c_i'(t)$ is guaranteed to stay within Ω.

Next, consider the $\dot{z}$-component of $c_i'(t)$. Using the collision law (5), the difference between the post collision $\dot{z}$ components under plastic and non-plastic collision is given by $z_1'(t_1) - \dot{z}_1 = ev_1\eta(z_1)$. For $t \in I_i'$, $i \geq 1$, the change in $\dot{z}_i'(t)$ can be decomposed into its discrete and continuous parts as:

$$\Delta_i^{(1)} = \dot{z}_i'(t_i') - \dot{z}_{i-1}'(t_i'), \qquad \Delta_i^{(2)} = \dot{z}_i'(t_{i+1}') - \dot{z}_i'(t_i').$$

According to the collision law (5), we have the following bounds on the discrete and continuous parts:

$$\|\Delta_i^{(1)}\| \leq (1+e)\eta_{max} v_i, \qquad \|\Delta_i^{(2)}\| \leq \ddot{z}_{max}\tau_i.$$

Using the triangle inequality and the bounds on τ_i and v_i in (23) and (21), the change in $\dot{z}_i'(t)$ can be bounded by:

$$\begin{aligned}\|\dot{z}_i'(t) - z_1\| &= ev_1\eta(z_1) + \sum_{i=2}^{\infty} \|\Delta_i^{(1)}\| + \sum_{i=1}^{\infty} \|\Delta_i^{(2)}\| \\ &\leq \left(\frac{1+\delta}{1-\delta e}\eta_{max} + \frac{2}{a_{min}(1-\delta^2 e)}\ddot{z}_{max}\right) e\kappa v^*.\end{aligned}$$

Applying the triangle inequality once again, condition (17) then implies that

$$\|\dot{z}_i'(t) - \dot{z}^*\| \leq \|\dot{z}_i'(t) - \dot{z}_1\| + \|\dot{z}_1 - \dot{z}^*\| \leq (1-\gamma)\epsilon_2 + \gamma\epsilon_2 = \epsilon_2,$$

and the $\dot{z}$-component of $c_i'(t)$ is guaranteed to stay within Ω. Finally, inequalities (25) and (22) along with condition (18) guarantee that $h_i'(t) \leq \epsilon_3$ and $\dot{h}_i'(t) \leq \epsilon_4$ for all $i \geq 1$, thus the h and $\dot{h}$-components of the solution also stay within Ω.

Step 3: We now use the results of Lemma 1 and Lemma 2 to prove the main result.

Proof (of Theorem 1). Consider the completed execution $\overline{\chi}' = \{\tilde{\chi}', \chi', ...\}$ with $e > 0$, under initial conditions $x_0 \in \Omega_2(\epsilon_1, \epsilon_2)$. Lemma 1 and Lemma 2 imply that χ' is a Zeno execution which reaches $\mathcal{S}$ in time t_∞, and that $c_i'(t) \in \Omega$ for all $i \geq 1$. Define the function $\Phi: \ \Omega_2(\epsilon_1, \epsilon_2) \to \mathcal{S}$ as

$$\Phi(x_0) = c_\infty', \text{ under initial condition } c_0'(0) = x_0.$$

Note that Φ is well-defined, since for initial conditions within Ω_2, a Zeno execution is guaranteed. Moreover, since the limit point satisfies $c_\infty' \in \Omega \cap \mathcal{S} = \Omega_2(\epsilon_1, \epsilon_2)$, Φ maps $\Omega_2(\epsilon_1, \epsilon_2)$ onto itself. The continuity of the hybrid flow with respect to its initial condition, which is a fundamental property of a completed hybrid system with a single constraint (cf. [5]) implies that Φ is continuous. Applying the fixed point theorem (cf. [12]), we conclude that there exists a fixed point $\bar{x} \in \Omega_2(\epsilon_1, \epsilon_2)$ such that $\Phi(\bar{x}) = \bar{x}$. Finally, the definition of Φ then implies that $\bar{x}$ corresponds to the starting point of a Zeno periodic orbit with period $T' = t_\infty'$.

5 Simulation Example

This section demonstrates the theoretical results on a simulation example of a mechanical system. The mechanical system under consideration is a constrained

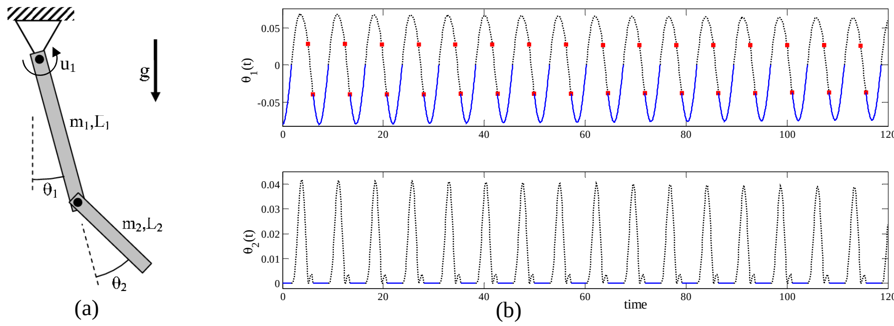

Fig. 2. (a) The constrained double pendulum system (b) Time plots of the solution $\theta_1(t)$ and $\theta_2(t)$ of the double pendulum with no actuation under plastic collisions.

double pendulum, which is depicted in Figure 2(a). The double pendulum consists of two rigid links of masses m_1, m_2, lengths L_1, L_2, and uniform mass distribution, which are attached by revolute joints, while a mechanical stop dictates the range of motion of the second link. The upper joint is actuated by a torque u_1 according to a control law $u_1 = u_1(\theta_1, \dot{\theta}_1)$, while the second joint is passive. This example serves as a simplified model of a leg with a passive knee and a mechanical stop, which is widely investigated in the robotics literature in the context of passive dynamics of bipedal walkers (cf. [15] and [21]).

The configuration of the double pendulum is $q = (\theta_1, \theta_2)$, and the constraint that represents the mechanical stop is given by $h(q) = \theta_2 \geq 0$. Note that in that case the coordinates are already in the form $q = (z, h)$, where $z = \theta_1$ and $h = \theta_2$. The Lagrangian of the system is given by

$$L(q, \dot{q}) = \tfrac{1}{2}\dot{q}^T M(q)\dot{q} + (\tfrac{1}{2}m_1 L_1 + m_2 L_1)g\cos\theta_1 + \tfrac{1}{2}m_2 L_2 g\cos(\theta_1 + \theta_2), \quad (26)$$

with the elements of the 2×2 inertia matrix $M(q)$ given by

$$\begin{aligned}
M_{11}(q) &= m_1 L_1^2/3 + m_2(L_1^2 + L_2^2/3 + L_1 L_2 \cos\theta_2)\\
M_{12}(q) &= M_{21} = m_2(3L_1 L_2 \cos\theta_2 + 2L_2^2)/6\\
M_{22}(q) &= m_2 L_2^2/3.
\end{aligned}$$

The unconstrained equations of motion take the form (2) with the control $u = (u_1(\theta_1, \dot{\theta}_1), 0)^T$. From this data we obtain the completed hybrid system $\overline{\mathscr{H}}_{\mathbf{L}}$.

The first running simulation shows the motion of the *uncontrolled system* i.e. $u_1 = 0$, under *plastic collisions*, i.e. $e = 0$. Fig. 2(b) shows the time plots of $\theta_1(t)$ and $\theta_2(t)$ under initial condition $q(0) = (-0.08, 0)$ and $\dot{q}(0) = (0, 0)$. The parts of unconstrained motion appear as dashed curves, and the parts of constrained motion appear as solid curves. The points of collision events are marked with squares ('■') on the curve of $\theta_1(t)$. The double pendulum exhibits a slightly decaying periodic-like motion with *two* plastic collisions per cycle. At each cycle, after the first plastic collision, the constraint force λ required to maintain the constraint $\theta_2 = 0$ is negative. Thus, the second link instantaneously detaches for

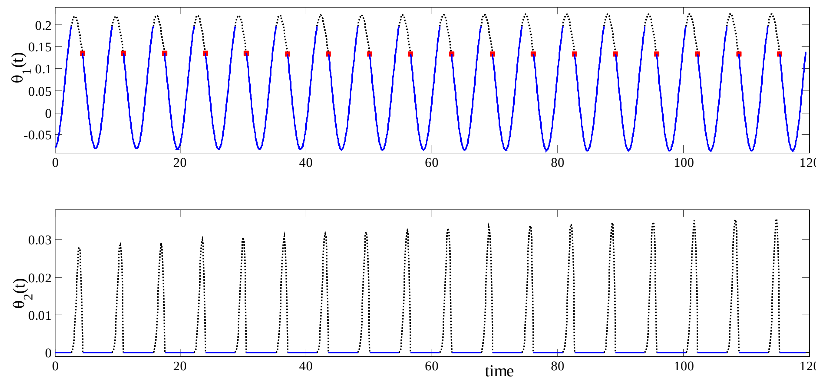

Fig. 3. Time plots of the solution $\theta_1(t)$ and $\theta_2(t)$ of the controlled double pendulum under plastic collisions.

another phase of unconstrained motion, until a second plastic collision occurs. After the second collision, the second link sticks, $\theta_2 = 0$, and the pendulum switches to a constrained motion with positive constraint force $\lambda > 0$ for some finite time, until λ crosses zero, and the second link detaches again.

In order to obtain a non-decaying periodic solution with a *single* plastic collision per cycle, i.e., a simple periodic orbit, we add a torque at the base of the first link which obeys the PD control law $u_1(\theta_1, \dot{\theta}_1) = -k_1(\theta_1 - \theta_{1e}) - c_1\dot{\theta}_1$. The control parameters are chosen as $k_1 = 0.5$, $\theta_{1e} = \pi/9$ and $c_1 = -0.01$. The proportional term associated with k_1 was chosen as to increase the positive acceleration $\ddot{\theta}_1$ and decrease the negative acceleration $\ddot{\theta}_2$ for $\theta_1 < 0$, and thus increase the constraint force λ that ensures that after the first collision, the second link does not detach. The negative dissipation term associated with c_1 injects a small amount of energy to the system that compensates for the losses due to collisions. Figure 3 shows the simulation results of the controlled double pendulum with plastic collisions under the same initial condition as above. One can clearly see convergence to a simple periodic orbit with a single plastic collision per cycle.

Next, we apply Theorem 1 to check for existence of a Zeno periodic orbit with $e > 0$. One can verify numerically that the assumptions of the theorem are satisfied and that, in particular, the simple periodic orbit obtained through the control law is locally exponentially stable with $\gamma = 0.9404$. Choosing $\epsilon_1 = 0.01$ and $\epsilon_2 = \epsilon_3 = 0.05$, Theorem 1 implies that if $e \leq 0.0014$, then the existence of a Zeno periodic orbit with initial conditions within $\omega_2(\epsilon_1, \epsilon_2)$ is guaranteed. Simulation of the double-pendulum system with $e = 0.0014$ verifies the convergence to a stable Zeno periodic orbit; it can be verified numerically that in fact the orbit is stable. The simulation results are not shown here, since they are not visually distinguishable from the results with $e = 0$.

In order to illustrate the strong conservatism of Theorem 1, we have conducted another simulation under the same initial condition as above, with a coefficient of restitution $e = 0.5$, which is much larger than the theoretical bound in the sufficient conditions (14)-(18). The infinite Zeno executions were truncated

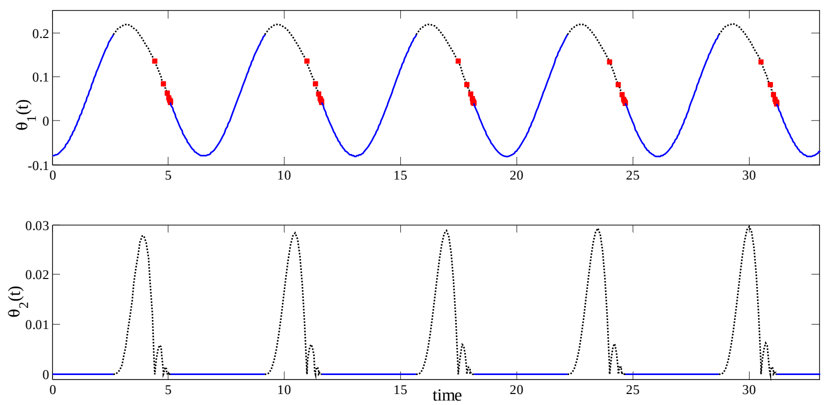

Fig. 4. Time plots of the solution $\theta_1(t)$ and $\theta_2(t)$ of the controlled double pendulum with $e = 0.5$.

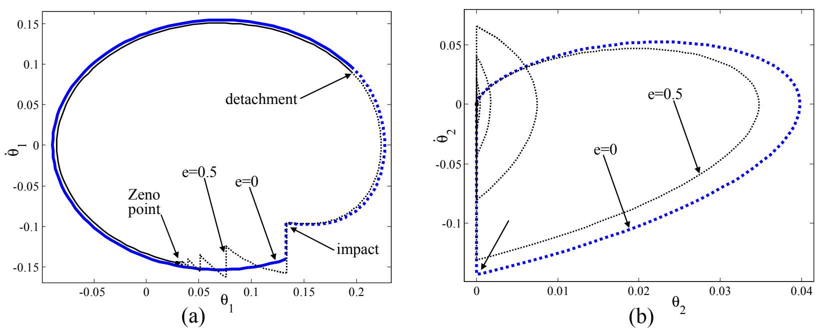

Fig. 5. Phase portraits of the periodic orbit in (a)$(\theta_1, \dot{\theta}_1)$-plane and (b)$(\theta_2, \dot{\theta}_2)$- plane for $e = 0$ (thin black) and $e = 0.5$ (thick blue)

after a finite number of collisions at which the collision velocity $\dot{h}$ is below a threshold of 10^{-10}. The simulation results, which are shown if Figure 4, clearly indicate the existence of a Zeno hybrid periodic orbit. Moreover, it can be verified numerically, that this orbit is also locally exponentially stable.

Figures 5(a) and 5(b) show the phase portraits of the periodic orbits in $(\theta_1, \dot{\theta}_1)$- and $(\theta_2, \dot{\theta}_2)$- planes, respectively, for coefficients of restitution $e = 0$ and $e = 0.5$. The direction of motion along the orbits in forward time is clockwise. The thick (blue) curves correspond to the case $e = 0$, and the thin (black) curves correspond to the case $e = 0.5$. The parts of unconstrained motion appear as dashed curves, and the parts of constrained motion appear as solid curves. Note that in Figure 5(b), the constrained motion collapses to the single point $(\theta_2, \dot{\theta}_2) = 0$. From the figures, one can clearly see how the nominal periodic orbit is perturbed under non-plastic collisions.

Finally, we have gradually increased the coefficient of restitution e and numerically checked for existence of Zeno periodic orbits. The largest value of e for which we obtained such an orbit was $e = 0.9225$. For this value of e, the duration of the constrained motion in the Zeno periodic orbit is very short, as shown in the simulation results of Figure 6. For larger values of e, *the phase of constrained motion vanishes*, and the execution is no longer Zeno. This transition can be viewed as a new type of bifurcation in Lagrangian hybrid systems, which we call a *Zeno bifurcation*, in which a Zeno periodic orbit ceases to be Zeno.

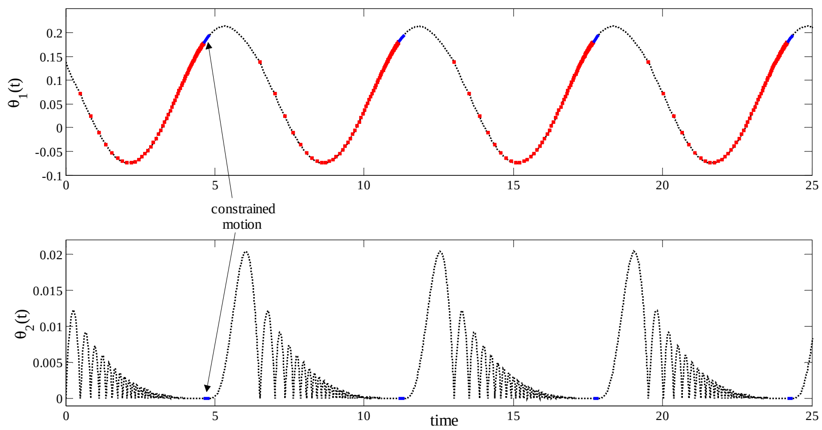

Fig. 6. Time plots of the solution $\theta_1(t)$ and $\theta_2(t)$ of the controlled double pendulum with $e = 0.9225$.

6 Conclusion

This paper considered two types of periodic orbits in completed Lagrangian hybrid systems: *simple* and *Zeno*. The main result presented is sufficient conditions on when a simple periodic orbit in a Lagrangian hybrid system implies the existence of a Zeno periodic orbit in the same Lagrangian hybrid system with a different coefficient of restitution. Moreover, these conditions give an explicit upper bound the change in the coefficient of restitution that guarantees existence of the Zeno periodic orbit.

These results indicate two major future research directions: better bounds on the allowable change in the coefficient of restitution and conditions on the preservation of stability. For the first direction, as was illustrated by the example, the obtained bounds are strongly conservative; extending these bounds in a rigorous fashion will be practically useful and theoretically satisfying. The second future research direction—studying stability—is even more interesting. The authors have been able to show that under certain simplifying assumptions, stability of the simple periodic orbit directly implies the stability of the Zeno periodic orbit. However, this preliminary result was not included in the paper due to space constraints. In the future, understanding how stability extends from one type of orbit to the other with the fewest possible assumptions will provide new and interesting challenges. Finally, extending the results to Lagrangian hybrid system *with multiple constraints* will enable the analysis of full models of bipeds with knees for designing stable walking and running under non-plastic collisions.

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A Proof of Lemma 1

In order to prove Lemma 1, we will utilize methods from optimal control. (This idea of was also used in [19] for proving stability of Zeno equilibria, and in [14] for stability analysis of differential inclusions.) We, therefore, briefly review the basic form of Pontryagin's maximum principle based on its presentation in [3], though we adopt a slightly different notation.

Consider a control system

$$\dot{x} = f(x, u), \tag{27}$$

where $x \in I\!R^n$ and $u \in U \subseteq I\!R^m$, where U is a convex set of admissible controls. A solution to (27) on a time interval $[t_0, t_f]$ is a pair $(x(t), u(t))$ satisfying (27) and $u(t) \in U$ for all $t \in [t_0, t_f]$; the initial and final conditions of $x(t)$ are denoted $x_0 = x(t_0)$ and $x_f = x(t_f)$. The design goal is to find a solution to (27) that minimizes a given cost function $P(x_f, t_f)$ [5]; note that the end condition x_f and the end time t_f, may be either specified or "free".

Using calculus of variations techniques, the solution of this problem is given as follows. First, define the *Hamiltonian*, as $H(x, u, \lambda, t) = \lambda(t)^T f(x, u)$, where $\lambda \in I\!R^n$ is called *the co-state vector*. The co-state dynamic equations are then given by $\dot{\lambda} = -\frac{\partial H}{\partial x}$, and the optimal control satisfies $u^*(t) = \mathsf{argmin} H$. The end condition is given by $[\frac{\partial P}{\partial x_f} - \lambda(t_f)]^T \delta x_f = 0$, where if a particular state variable x_i is specified, then its variation $\delta x_i(t_f)$ vanishes, and if it is not specified, then it gives an end condition for the corresponding co-state variable $\lambda_i(t_f)$. In case where the terminal time t_f is not specified, an additional condition on $H(t_f)$ is given by $\frac{\partial P}{\partial t_f} + H(t_f) = 0$.

Proof (of Lemma 1). We begin by proving (21). Consider a time interval I_i of the hybrid execution χ'. Our goal is to find bound on the ratio v_{i+1}/v_i. Choosing

[5] Many textbooks also consider an integral cost function of the form $J = \int_{t_0}^{t_f} g(x, u, t)dt$. This cost function can be incorporated into the formulation here by using an additional state variable y, whose dynamics is given by $\dot{y} = g(x, u, t)$. The cost function is then simply given by $P = y(t_f)$.

a state vector $x = (x_1, x_2) = (h_i'(t), \dot{h}_i'(t))$, its dynamics under the differential inclusion (19) can be formulated as a control system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned} \tag{28}$$

where $u \in [-a_{max}, -a_{min}]$.

Setting the initial time $t_0 = t_i$, the initial conditions are $x_1(t_0) = 0$, and $x_2(t_0) = v > 0$. Next, consider the cost function: $P(x_f, t_f) = x_2(t_f)$ for the control system (28). The Hamiltonian is given by $H = \lambda_1 x_2 + \lambda_2 u$. The co-state dynamic equations are then $\dot{\lambda}_1 = 0$ and $\dot{\lambda}_2 = \lambda_1$, indicating that $\lambda_1(t)$ is constant and $\lambda_1(t)$ is a linear function. The end condition gives $\lambda_2(t) = 1$. The maximum principle then implies that the optimal input $u^*(t)$ is either a_{min} or a_{max}, and depends solely on the sign of $\lambda_2(t)$, which is a linear function that has at most one zero-crossing point. Therefore, u^* is a piecewise-constant function with at most one switching point, and we can set $u^*(t) = -u_1$ for $t \in [t_0, t_s]$ and $u^*(t) = -u_2$ for $t \in [t_s, t_f]$, where t_s is the unknown switching time, and $u_1, u_2 \in \{a_{max}, a_{min}\}$. Using this information, one can find the optimal solution for $x_2(t_f)$ by direct integration of (28), which gives

$$x_2(t_f) = -\sqrt{(v - u_1 t_s)^2 + 2u_2(v t_s - u_1 t_s^2/2)},$$

whose critical value is attained at $t_s^* = v/u_1$, i.e. it satisfies $x_2(t_s) = 0$. It then follows that the minimal value of $x_2(t_f)$ is obtained for $u_1 = a_{min}$ and $u_2 = a_{max}$ as $x_2^*(t_f) \geq -\delta v$, where $\delta = \frac{a_{max}}{a_{min}}$. Using (20) along with the definition of v_i and v_{i+1}, one gets that $v = e v_i$, $x_2(t_f) = -v_{i+1}$. The optimal solution then implies that $v_{i+1} =\leq e\delta v_i$. Recall that condition (15) implies that $e\delta < 1$. Thus, the series of v_i is bounded by a decaying geometric series with factor $e\delta$, which completes the proof of (21). Next, since $\ddot{h}_i'(t_i) < 0$, $\dot{h}_i'(t_i)$ is monotonously decreasing, and the condition $e\delta < 1$ implies that $|\dot{h}_i'(t_i)| \leq v_i < v_1$, which proves (22).

In order to prove (23), we need to establish an upper bound on τ_{i+1}/τ_i. Consider the differential inclusion (19) for two consecutive time intervals $I_{i-1} = [\tau_{i-1}, \tau_i]$ and $I_i = [\tau_i, \tau_{i+1}]$. That is, we consider two control systems as defined in (28). The initial conditions and final conditions for the first control system are given by:

$$\begin{aligned} x_1(t_{i-1}') &= h_{i-1}'(t_{i-1}') = 0. \\ x_2(t_{i-1}') &= \dot{h}_{i-1}'(t_{i-1}') = e v_{i-1}, \\ x_1(t_i') &= h_i'(t_i') = 0. \\ x_2(t_i') &= \dot{h}_{i-1}'(t_i') = -v_i \end{aligned} \tag{29}$$

where v_i and t'_i are not specified. The initial and final conditions for the second control system are:

$$x_1(t'_i) = h'_i(t'_i) = 0.$$
$$x_2(t'_i) = \dot{h}'_i(t'_i) = ev_i$$
$$x_1(t'_{i+1}) = h'_(t'_{i+1}) = 0.$$
$$x_2(t'_{i+1}) = \dot{h}'_i(t'_{i+1}) = v_{i+1}$$

where t'_{i+1} and v_{i+1} are not specified.

The goal is to find a solution to the two control systems in which τ_{i+1}/τ_i is maximized where, again, $\tau_i = t'_{i+1} - t'_i$. It is easy to see under a given initial condition for the second control system, τ_{i+1} is maximized by taking $u(t) = -a_{min}$ for $t \in I_i$, and its maximum value is given by $\tau^*_{i+1} = 2ev_i/a_{min}$. The problem then reduces to maximizing the ratio v_i/τ_i for a solution to the first control system. The definition of the Hamiltonian H and the derivation of the co-state dynamic equation for $\lambda(t)$ are also identical to those derived in the proof of (21). Setting $t_0 = t'_{i-1} = 0$, the cost function to be minimized in this problem is given by $P(x_f, t_f) = x_2(t_f)/t_f$ where here $t_f = t'_i$. As before, the maximum principle implies that the optimal input $u^*(t)$ is either a_{min} or a_{max}, and depends solely on the sign of $\lambda_2(t)$. Using the end condition for λ_2 gives $\lambda_2(t_f) = 1/t_f$, which implies that $\lambda_1(t) = c_1$ and $\lambda_2(t) = 1/t_f + c_1(t_f - t)$. The additional condition on $H(t_f)$ gives $x_2(t_f)(c_1 - 1/t_f^2) + u(t_f)/t_f = 0$. Since $x_2(t_f)$ and $u(t_f)$ are both negative, we conclude that $c_1 - 1/t_f^2 < 0$. This implies that $\lambda_2(t)$ does not cross zero, and is positive for $t \in [0, t_f]$. Therefore, minimization of the cost function is obtained by taking the *constant* input $u(t) = -a_{max}$ for $t \in I_i$, and the maximum value for τ_{i+1}/τ_i is consequently $e\frac{a_{max}}{a_{min}} = e\delta^2$. Condition (15) then implies that the series of v_i is bounded by a decaying geometric series with factor $e\delta^2 < 1$, hence $\tau_i \leq \tau_1(e\delta^2)^{i-1}$. Finally, considering the differential inclusion (19) for the time interval I_1, it is easy to see that $\tau_1 \leq 2v_1/a_{min}$, which completes the proof of (23).

The proof of (24) then follows directly, since the Zeno time is obtained as the sum $t'_\infty - t'_1 = \sum_{i=1}^{\infty} \tau_i$, which is bounded by the sum of the geometric series in (23).

In order to prove (25), note that $h'_i(t)$ attains its maximum value in I_i at the time t_m that satisfies $\dot{h}'_i(t_m) = 0$. Is is straightforward to show (even without formulating an optimal control problem) that $h'_i(t_m)$ is maximized with $u = -a_{min}$. Therefore, one gets $h_i(t) \leq 2(\dot{h}'_i(t'_i))^2/a_{min} = 2e^2v_i^2$ for all $t \in I_i$, and (25) follows directly from (21)