

Experimental Demonstration of the Dynamics and Stability of a Low Reynolds Number Swimmer Near a Plane Wall

Sebastian Zhang, Yizhar Or, and Richard M. Murray

Abstract— The motion of microorganisms as well as of tiny robotic swimmers for biomedical applications is governed by low Reynolds number (Re) hydrodynamics, where viscous effects dominate and inertial effects are negligible. This paper presents experimental results that verify theoretical predictions of our recent work which analyzed the dynamics and stability of a low- Re swimmer near a plane wall. The experimental setup uses macro-scale swimmer prototypes which are propelled by rotating cylinders in highly viscous silicone oil. The motion is recorded by a video camera and position measurements are taken by an optical tracking system. The results show good qualitative agreement with our recent theoretical predictions.

I. INTRODUCTION

The motion of microorganisms [1], as well as of tiny robotic swimmers for biomedical applications [2], is governed by low Reynolds number hydrodynamics [3,4]. The Reynolds number, which encompasses the ratio of inertial forces to viscous forces, is defined as $Re=VL/\nu$, where V is a characteristic velocity, L is a characteristic length scale, and ν is the kinematic viscosity of the fluid. For example, a typical Reynolds number for a human swimmer who is governed by inertial effects is in the order of 10^4 , whereas a microorganism typically has $Re\approx 10^{-3}$ and moves by harnessing viscous effects. In this paper, we report experimental results that verify theoretical predictions of our previous work [5], which analyzed the dynamics and stability of a low- Re swimmer near a plane wall.

The theory of low- Re swimming has been widely studied in the physics and biology literature [1,3]. In the applicative direction, some efforts to develop miniaturized artificial swimmers were reported, e.g. [2,6-8]. One typical strategy for low- Re swimming is applying cyclic changes in the internal configuration of the swimmer [9-11]. Another strategy is applying purely tangential surface deformations such that the shape of the swimmer remains constant [12,13]. It was only recently that this problem has been studied from a control-theoretic viewpoint [14,15]. Earlier

works considered geometric-mechanical aspects of swimming in both physics [16,17] and dynamical control systems literature [18-20]. All the works mentioned above used the assumption that the fluid domain is unbounded. Under this simplification, the dynamics of the swimmer enjoys a special geometric structure of a connection on a fiber bundle. However, the presence of solid boundaries destroys this structure, and the interaction between the swimmer and the boundary becomes much more complicated to model [21].

Several works have studied wall-swimmer interaction in the context of microorganism motility. Some of them presented numerical simulation results [22,23], and some reported experimental observations [24,25] and suggested simplified theoretical models [25,26]. However, none of these works analyzed the problem from control-theoretic or geometric-mechanical viewpoint. A major step in these directions has been conducted in our recent work [5]. In this work, we studied the dynamics of low- Re swimming near an infinite plane wall for simple articulated swimmers composed of a rigid structure of spheres which are rotated about their axis in order to generate propulsion. We have shown that a swimmer possessing fore-aft symmetry is able to translate steadily parallel to the wall. When this steady motion is perturbed, the swimmer exhibits neutrally stable oscillatory motion. Moreover, if the fore-aft symmetry of the swimmer is broken, it is possible to achieve open-loop asymptotic stability of the parallel translation without requiring sensing or feedback.

The goal of this paper is to verify the theoretical predictions of [5] by presenting results of motion experiments conducted with macro-scale swimmer prototypes in a highly viscous fluid. The swimmers are propelled by rotating cylinders using DC motors. Planar motion of the swimmer is enforced by a flotation structure that supports the gravitational load. The motion experiments were recorded by a video camera (movies are available online at [27]), and position measurements were taken using an optical tracking system. The results show good qualitative agreement with the theory in [5] and corroborate our predictions on the physical swimmer-wall interaction.

The structure of the paper is as follows. Section II summarizes the theoretical background presented in [5]. Section III details the experimental methods and the different swimmers used. Section IV presents the experimental results. Section V discusses the results. Conclusions and future work are summarized in section VI.

S. Zhang is with the Mechanical Engineering Department, Clemson University, SC 29634 USA (phone: 864-908-6466; e-mail: sebastz@g.clemson.edu).

Y. Or is with the Mechanical Engineering Department, Technion – Israel Institute of Technology, Haifa 32000, Israel (e-mail: izi@tx.technion.ac.il).

R. M. Murray is with the Control and Dynamical Systems Department, California Institute of Technology, CA 91125 USA, (e-mail: murray@cds.caltech.edu).

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II. THEORETICAL BACKGROUND

We now give background on the theoretical model of swimmer studied in [5], and review some geometric properties of the resulting dynamics.

A. Theoretical swimmer model

Consider a simplified model of an articulated micro-swimmer which consists of a collection of n rigid spheres of equal radius a . The spheres are attached to a thin rigid structure, called the *body* of the swimmer. It is assumed that the motion of the swimmer is restricted to the xy plane. The swimmer is submerged in a viscous fluid which is bounded by an infinite solid wall at the plane $y = 0$. The generalized coordinates describing the position and orientation of the swimmer in the plane are denoted $q = (x, y, \theta) \in SO(2)$. Some of the spheres are actuated by rotation about their z -axis which is fixed to the body. The *input* for the swimmer $u \in \mathcal{R}^m$ is the vector of the angular velocities of all actuated spheres with respect to the body. The simplest example is the *two-sphere swimmer* shown in Fig. 1(a), in which two spheres are connected by a thin rigid rod of length d , and are actuated by angular velocities $u=(u_1, u_2)$. The second example is the *2+1-sphere swimmer* shown in Fig. 1(b), in which an additional unactuated sphere is rigidly attached to the body.

The Reynolds number governing the motion is assumed to be negligibly small, so that the fluid velocity satisfies the *Stokes equation* [4], which is the limit of Navier-Stokes equation for the case $Re \rightarrow 0$. The boundary conditions are called *no-slip conditions*, and imply that the fluid velocity matches the velocity of the spheres at their boundaries, and vanishes on the stationary wall. The motion is assumed to be *quasistatic*, so that the fluid exerts zero net force and torque on the swimmer at all times. These assumptions lead to complicated dynamic interaction between the fluid and the swimmer. Nevertheless, a fundamental property of Stokes flow around solid bodies is the existence of a *linear relation* between the velocities of all bodies and the forces exerted by the fluid on the bodies. This relation is encapsulated in a matrix called the *resistance matrix*, which depends nonlinearly on the configuration of the bodies, and cannot, in general, be computed exactly. In [5], the dynamics of the swimmer is formulated by using approximate expressions for the resistance matrix, which were developed by Swan and Brady [28] for the case of multiple spheres interacting with a plane wall. It is shown in [5] that the of motion of the swimmer under given input $u(t)$ is governed by a *driftless nonlinear control system* of the form

$$\dot{q} = G(q)u. \quad (1)$$

The matrix $G(q)$ depends nonlinearly on the configuration q and encapsulates the hydrodynamic interaction between the swimmer, the fluid, and the wall.

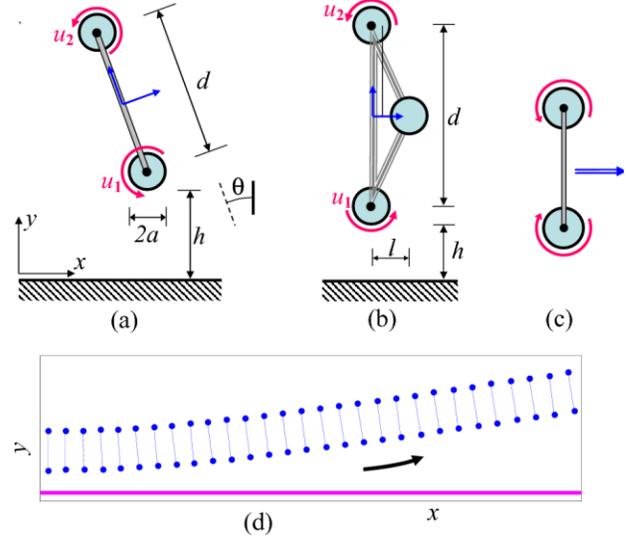


Fig. 1. (a) The two-sphere swimmer. (b) The 2+1-sphere swimmer. (c) The two-sphere swimmer with $u_1 = -u_2$ in unbounded fluid. (d) Snapshots of the two-sphere swimmer with $u_1 = -u_2$ near a wall

B. Geometric Properties of the Equation of Motion

We now review some basic geometric properties of the equation of motion (1) for our system. First, consider the two-sphere swimmer under input $u_1 = -u_2$, i.e. opposite and equal angular velocities of the spheres. When placed in unbounded fluid, the swimmer will move in pure translation along its symmetry axis as shown in Fig. 1(c). However, the presence of the boundary breaks the symmetry, and under the same input, the swimmer will rotate and deviate away from the wall (Fig. 1(d)). Thus, we focus here on achieving swimming motion of *pure translation parallel to the wall*.

An important observation is that the equation of motion (1) is invariant under translation parallel to the wall, in the x -direction. That is, $G(q)$ in (1) depends only on y and θ , and the dynamics of y and θ is thus decoupled from that of x . Therefore, denoting $q' = (y, \theta)$, the dynamics of q' is governed by the equation

$$\dot{q}' = G'(q')u, \quad (2)$$

where G' is the 2×2 lower submatrix of G in (1). Under this setup, pure translation parallel to the wall corresponds precisely to *equilibrium solutions* of (2).

A second important observation in [5] focuses on the case of the two-sphere swimmer, which possesses *fore-aft symmetry*. In this case, (2) has a special structure called *reversing symmetry*, which implies that elements of the first line of $G'(q')$ are even functions of θ and elements of the second line of $G'(q')$ are odd functions of θ . Thus, for each distance from the wall $y = y_e$, there exists a constant input u_e with a specific ratio u_1/u_2 such that $q' = (y_e, 0)$ is an equilibrium point of (2) under the input $u = u_e$, which corresponds to swimming parallel to the wall with the orientation $\theta = 0$. Moreover, it is shown in [5] that if y_e is greater than a lower bound, then the linearization of (2) about $(y_e, 0)$ has a pair of purely imaginary eigenvalues. In that case, $(y_e, 0)$ is called a *reversible Lyapunov center*,

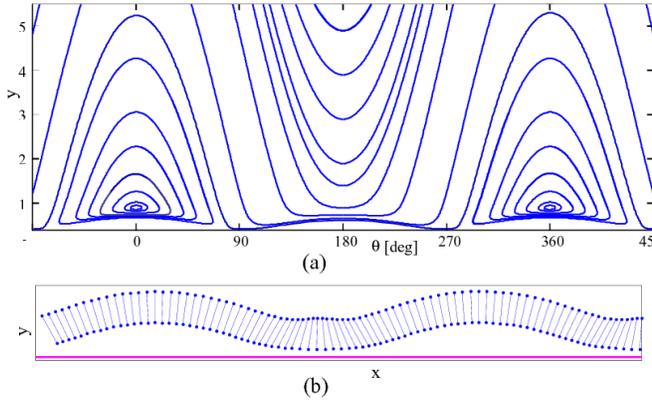


Fig. 2. (a) Phase portrait in (y, θ) -plane for the two-sphere swimmer near a wall under constant input. (b) Snapshots of the swimmer's periodic motion which is a marginally stable equilibrium point enclosed by a continuum of periodic orbits. Fig. 2(a) depicts the phase portrait of trajectories in (y, θ) -plane under a fixed input u_c . Any small initial perturbation about $(y_e, 0)$ results in periodic (y, θ) -trajectory, which corresponds to an oscillatory motion along the wall, as shown in the snapshots in Fig. 2(b).

Finally, [5] studies the case where the swimmer's fore-aft symmetry is slightly broken, as in the 2+1-sphere swimmer of Fig. 1(b). In this case, under a constant input $u = \pm u_c$, there exists an equilibrium point (y_e, θ_e) , with a shifted orientation $\theta_e \neq 0$. Moreover, for one of the choices of the sign of u_c , the equilibrium point becomes *open-loop asymptotically stable*. As an example, Fig. 3(a) depicts the phase portrait of the 2+1-sphere swimmer under a constant input, and Fig. 3(b) shows time plot of $\theta(t)$ converging to its nonzero equilibrium value. Fig. 3(c) shows snapshots of the motion of the swimmer. It can be seen that the perturbed swimmer converges passively to steady translation parallel to the wall.

III. EXPERIMENTAL METHODS

In this section, we describe the experimental setup which was used in order to demonstrate the theoretical results obtained in [5] with a macro-scale prototype. In order to keep the Reynolds number small in the macro-scaled system, the test fluid was chosen to be highly viscous silicone oil (Polydimethylsil-oxane). The kinematic viscosity of the silicone oil is $\nu = 60,000 \text{ cSt}$ with specific gravity of 0.976. The near unity value of specific gravity allows for preliminary tests of flotation to be conducted in water.

First, we constructed a swimmer prototype which is propelled by two rotating cylinders submerged in the fluid. The purpose of this system was to confirm theoretical results with a two-sphere model. The experiment was done with nylon cylinders (6.45cm length x 2.54cm diameter, 6cm center-center distance) instead of spheres in order to simplify the manufacturing and assembly. Even though the exact interaction of the swimmer with the fluid is different from the theoretical model, the geometric properties and qualitative results are preserved.

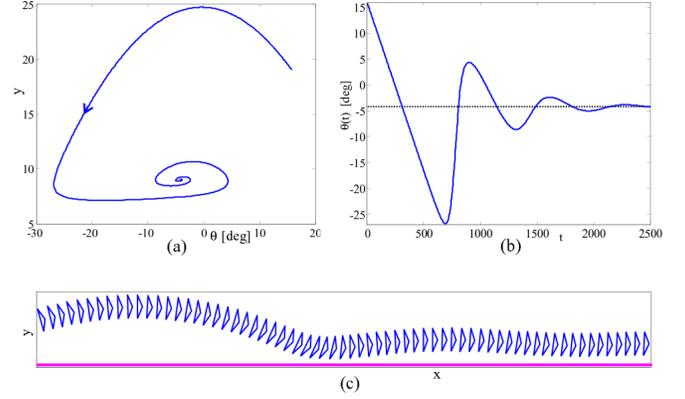


Fig. 3. (a) Phase portrait in (y, θ) -plane for the 2+1-sphere swimmer near a wall under constant input. (b) Time plot of $\theta(t)$. (c) Snapshots of the swimmer's motion.

The cylinders are rotated using Micromo 1319SR DC motors connected to a power supply. The angular velocity of each cylinder was proportional to the motor input voltage. In air, the average angular velocity was approximately 60 revolutions per minute at input voltage of 3V. In silicone oil, the average angular velocity was 30 revolutions per minute at 6V. The voltage-velocity ratio was reduced by a factor of four due to the resistance of the viscous fluid. The motors are attached to a wooden frame supported by foam cells floating on the fluid. The goal of the flotation system is to balance gravity and keep the motion of the swimmer two-dimensional in horizontal plane. The fluid is contained in a tank with dimensions of 110cm x 50cm x 17cm. The Reynolds number for this swimmer was calculated according to the formula $Re = \omega a d / \nu$ where ω is the angular velocity, a is the radius of the cylinders and d is the center-to-center distance. The characteristic Reynolds number is found to be $Re = 0.017$, which justifies the assumption of viscous flow. A picture of the swimmer is shown in Fig. 4.

An infrared Optitrack Flex V:100 camera was used for tracking. Code was written using the Optitrack SDK in C++ to track two reflective markers attached to the motors. LED's on the camera emit infrared light and the lens picks up the reflection. Suitable objects are determined through programmable threshold variables. These objects are then ranked based on programmable weighting variables. An extensive custom filtering algorithm was written to further erase noise and guarantee tracking of the two desired markers. The output is calibrated against two markers along the wall of the container serving as the reference for the x -axis. The sampling rate of the camera was 100 Hz but due to memory allocation limitations in C++, we took one out of every five data points (20 Hz).

Two experiments were conducted with this swimmer near the wall, as follows. In the first experiment, the two cylinders are rotating at equal and opposite angular velocities under input voltages of 6V. In the second experiment, the input voltages for the motor closer to the wall was 5V, while the input voltage to the other motor was 6V, giving unequal angular velocities of the cylinders.

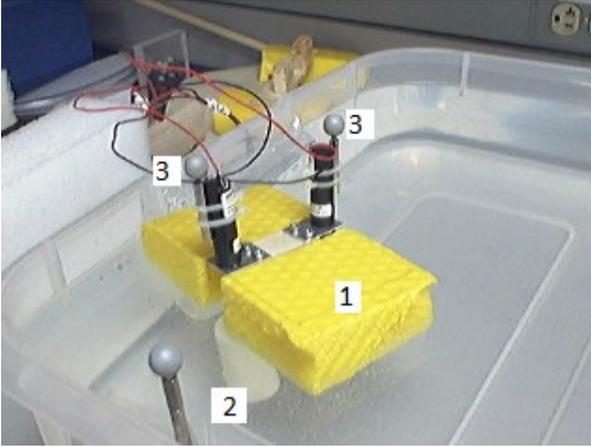


Fig. 4. Picture of 2-cylinder swimmer. 1. Foam Flotation
2. Nylon Cylinders (2.54cm diameter) 3. Motors / Tracking Markers

Measurements from the optical tracking system were recorded for both runs.

In order to demonstrate the theoretical predictions on the 2+1-sphere swimmer, another swimmer was constructed. This swimmer had further improvements for stability and rigidity with extra supports for the motors to maintain their vertical orientation. A third cylinder with diameter of 1cm was attached rigidly to the front of the swimmer. A picture of this swimmer is shown in Fig. 5. Experiments conducted with this swimmer include swimming near the wall with input voltages of 6V and 5V for which the cylinders are rotating at unequal angular velocities, under different initial distances from the wall.

IV. RESULTS

The raw measurements from the tracking system were noisy. We identified a clear source of the nearly periodic noise signal with frequency ~ 0.5 Hz, which is precisely the rotation frequency of the cylinders. We believe that the noise is a consequence of some small degree of eccentricity at the shafts and the cylinders caused by the set screws and additional misalignment, which excited slight lateral oscillations. In order to filter these noises out, all of the measured signals were filtered by a discrete-time low-pass Butterworth filter.

Using measurements for the two-cylinder swimmer, Fig. 6 shows snapshots of the motion in the xy plane near a wall at $y=0$ under equal and opposite angular velocities. It can be seen that the swimmer rotates and deviates away from the wall as predicted. Fig. 7 shows snapshots of the motion under unequal angular velocities. The result is a nearly periodic motion, as predicted by the theory. Fig. 8 shows phase portrait in (y,θ) plane for the two-cylinder swimmer near the wall under four different initial conditions with constant unequal input voltages. Note that the trajectories are slightly *diverging*, in contrast to the theoretical prediction of perfectly closed orbits as in Fig. 8. Possible reasons for this discrepancy are discussed in the next section. A movie of one representative run with nearly periodic motion can be

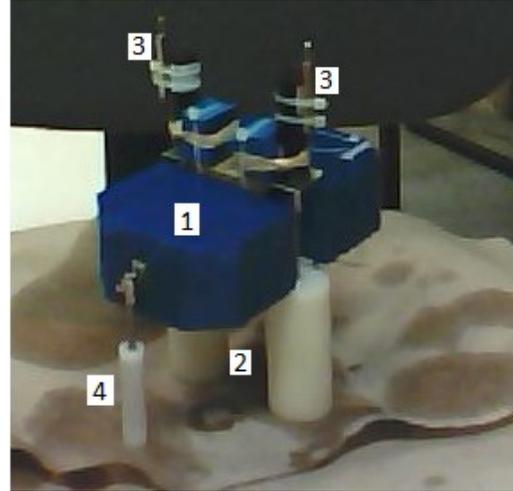


Fig. 5. Picture of swimmer with 2+1-cylinders 1. Foam Flotation
2. Nylon Cylinders (2.54cm diameter) 3. Motors / Tracking Markers
4. Nylon Cylinder (1cm diameter)

found in [27].

Fig. 9 shows phase portrait in (y,θ) plane for the 2+1-cylinder swimmer near the wall under eight different initial conditions with constant and unequal input voltages. The thick curve in the plot corresponds to a run where y and θ are almost constant, indicating the *existence of a stable equilibrium* of fixed distance from the wall and fixed nonzero angle. A movie of this run is available in [27]. When given an initial perturbation, the theory predicts asymptotic convergence towards the equilibrium point. Our experiments are conducted along a distance which is too short to demonstrate convergence. Nevertheless, using different initial conditions, the phase portrait in Fig. 8 clearly shows a tendency of the trajectories to spiral in towards an equilibrium point, which indicates asymptotic stability, as predicted by the theory and shown in Fig. 3(a).

V. DISCUSSION

The results display *qualitative agreement* with the behavior predicted in the theory. We now list some discrepancies between the experimental results and the theoretical predictions and suggest possible explanations.

First, note that trajectories of the two-cylinder swimmer with unequal angular velocities in Fig. 8 are slightly *diverging*, in contrast to the theoretical prediction of perfectly closed orbits as shown in Fig. 2(a). A possible explanation is that the exact periodic motion is a consequence of the swimmer being perfectly fore-aft symmetric in the horizontal plane. In the experimental system, misalignment of the flotation structure as well as small off-plane inclination of the swimmer during motion can add some degree of asymmetry which causes the slight divergence.

Second, in some of the trajectories in the phase portrait of Fig. 8 for the 2+1-cylinder swimmer, there is a noticeable transient in the beginning of each run until the trajectory is

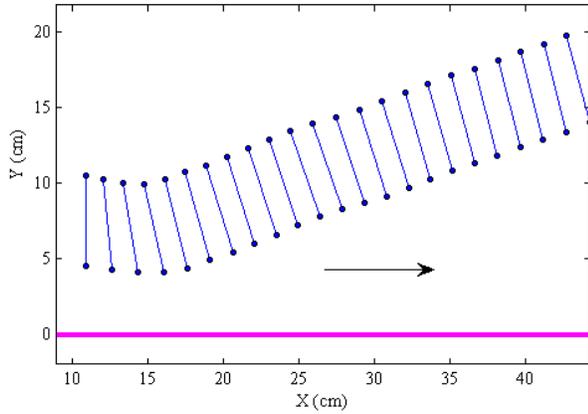


Fig. 6. Motion measurements of the 2-cylinder swimmer near the wall with equal motor voltages (wall is at $y=0$)

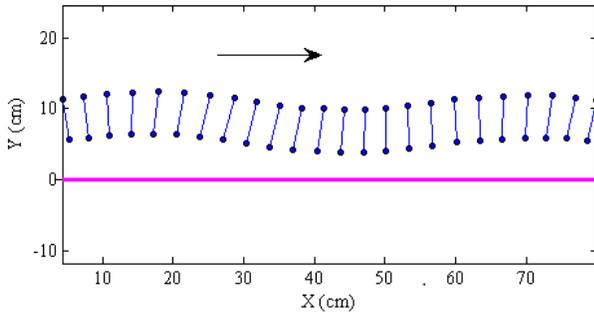


Fig. 7. Motion measurements of the 2-cylinder swimmer periodic motion near the wall with unequal motor voltages (wall is at $y=0$).

settled to a spiral curve. This transient can be explained by the duration where the DC power supply develops the desired input voltages once it is initially turned on.

Third, in the phase portraits of Figures 13 and 14 for the 2-cylinder and 2+1-cylinder swimmers, some trajectories are intersecting each other, in contrast to the theoretical prediction shown in Figures 2(a) and 3(a) and to the uniqueness of solutions for planar dynamical systems. This can be explained by unmodelled effects that induce higher-order dynamics on top of the simple planar system considered in the theoretical model. Examples of such unmodelled effects are off-plane dynamics of the swimmer and inertial effects due to the finite swimmer size.

Other possible sources for discrepancies in the repeatability of runs are associated with the power supply and the DC motors. The assumption that setting input voltages is equivalent to setting velocities is not completely precise due to the unmodelled dynamics of the motor's electric circuit. Moreover, significant heating of the wires was observed during experiments, which may have changed the voltage-velocity ratio and affected the repeatability of the results. Another possible reason for deviation of the actual input voltages from their desired values is the low resolution of the manual voltage control in the DC power supply. Finally, in several experimental runs, we observed events where one of the motors was stuck for a few seconds, possibly due to axis misalignment or internal shaft slippage, which could also explain some of the discrepancies.

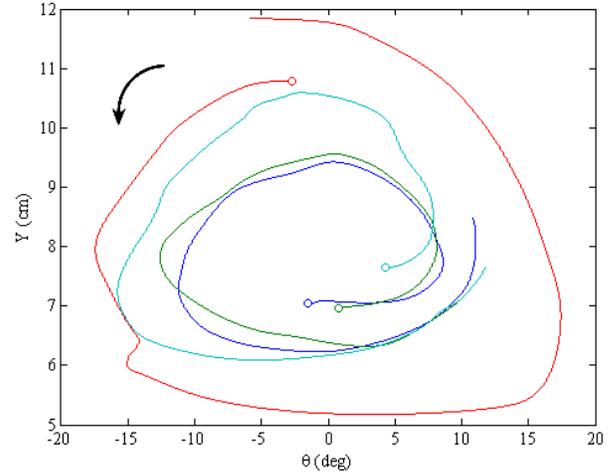


Fig. 8. Phase portraits of experiments with 2-cylinder swimmer. The circles represent the initial positions.

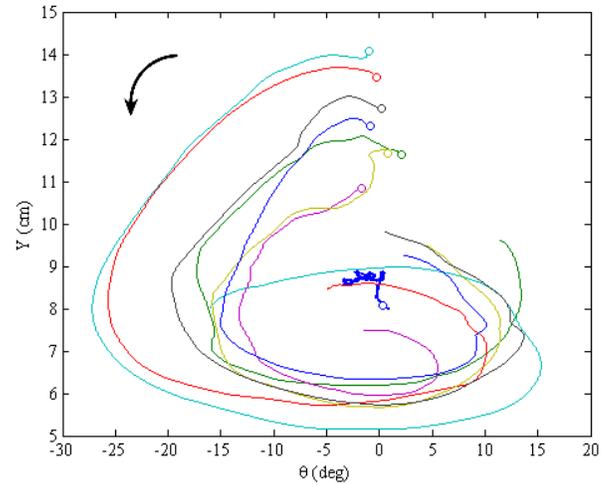


Fig. 9. Phase portraits of experiments with 2+1-cylinder swimmer. The circles represent the initial positions. The thick curve corresponds to a run where swimming was near parallel.

Since the two-cylinder swimmer did not produce as many wavelengths as we would have liked due to the physical limitation of the tank length, we now briefly discuss our attempts to make a down scaled swimmer. The behavior of the swimmer scales with the size of the cylinders: the smaller the cylinders, the smaller the wavelength of the periodic motion. We tried to construct smaller swimmers in order to observe more wavelengths in the same tank. The resulting swimmers did not work due to two main reasons. First, the amount of foam required did not scale down very much since we used the same motors, which required roughly the same area of foam for flotation. The submerged nylon cylinders were only slightly denser than the fluid so their effect on flotation is negligible. Second, there was tilting of the swimmer and evolution of off-plane motion near the wall, which led to tipping over of the swimmer or collision of the foam with the wall.

VI. CONCLUSION

In this paper, we reported motion experiments with macro-scale prototypes of low Reynolds number swimmers propelled by rotating cylinders which were supported by a flotation structure. The experiments reproduced the theoretical behavior predicted in our recent work [5]: When the angular velocities of the cylinders were equal, the swimmer rotated and diverged away from the wall; When the angular velocities of the cylinders were unequal, periodic motion near the wall was observed; A swimmer with broken fore-aft symmetry demonstrated convergent behavior to asymptotically stable translation parallel to the wall as predicted by the theory. Several attempts to construct a swimmer with smaller cylinders were not successful due to physical limitations and wall collisions caused by the flotation structure.

We now briefly discuss some possible directions for future extensions of the results. First, the finite length of the tank was a key limitation in observing long-distance convergence of the swimmer to a steady state motion. A possible solution is to conduct motion experiments in a *circular tank*, so that the swimmer can travel over an unlimited distance. This will require adaptation of the theory to circular walls, but analogous symmetry and reversibility arguments will also hold in that case. Second, in order to improve reliability and repeatability of the results, it is possible to add velocity control that will dictate the input angular velocities directly. This requires adding velocity sensors or encoders at each motor axis, which will affect the weight of the swimmer and the size of the required flotation structure. Third, the experiment enabled only *qualitative* comparison with the theory, since the model in [5] only considers rotating spheres and accounts for *far-field* interaction with the wall. Development of a more refined theoretical model which will enable *quantitative* comparison with experiments is currently an open problem. Finally, we plan to extend both theory and experiments to shape-changing swimmers, such as the classical example of Purcell's 3-link swimmer [3,9].

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