Existence of Periodic Orbits with Zeno Behavior in Completed Lagrangian Hybrid Systems

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Abstract. In this paper, we consider hybrid models of mechanical systems undergoing impacts — *Lagrangian hybrid systems*, and study their periodic orbits in the presence of *Zeno behavior*, where an infinite sequence of impacts converges in finite time. The main result of this paper is explicit conditions under which the existence of stable periodic orbits for a Lagrangian hybrid system with *perfectly plastic impacts* implies the existence of periodic orbits in the same system with *non-plastic impacts*. Such periodic orbits contain phases of constrained and unconstrained motion, and the transition between them necessarily involves Zeno behavior. The result is practically useful for a wide range of unilaterally constrained mechanical systems under cyclic motion, as demonstrated through the example of a double pendulum with a mechanical stop.

1 Introduction

Periodic orbits play a fundamental role in the design and analysis of hybrid systems modeling a myriad of applications ranging from biological systems to chemical processes to robotics [25]. To provide a concrete example, bipedal robots are naturally modeled by hybrid systems [8, 13]. The entire process of obtaining walking gaits for bipedal robots can be viewed simply as designing control laws that create stable periodic orbits in a specific hybrid system. This is a theme that is repeated throughout the various applications of hybrid systems [12].

In order to better understand the role that periodic orbits play in hybrid systems, we must first restrict our attention to hybrid systems that model a wide range of physical systems but are simple enough to be amenable to analysis. In this light, we consider *Lagrangian hybrid systems* modeling mechanical systems undergoing impacts; systems of this form have a rich history and are useful in a wide-variety of applications [5, 20, 26]. In particular, a *hybrid Lagrangian* consists of a configuration space, a Lagrangian modeling a mechanical systems, and a *unilateral constraint function* that gives the set of admissible configurations for this system. When the system's configuration reaches the boundary of its admissible region, the system undergoes an *impact event*, resulting in discontinuous velocity jump. The benefit of studying systems of this form is that they often display Zeno behavior (when an infinite number of impacts occur in a finite amount of time), so they give an ideal class of systems in which to gain an intuitive understanding of Zeno behavior and its relationship to periodic orbits in hybrid systems, which is the main focus of this paper.

Before discussing the type of periodic orbits that will be studied in this paper, we must first explain how one deals with Zeno behavior in Lagrangian hybrid systems by *completing* the hybrid model of these systems. Using the special structure of Lagrangian hybrid systems, the main observation is that points to which Zeno executions converge—*Zeno points*—must satisfy constraints imposed by the unilateral constraint function. These constraints are *holonomic* in nature, which implies that after the Zeno point, the hybrid system should switch to a holonomically constrained dynamical system evolving on the surface of zero level set of the constraint function. Moreover, if the force constraining the dynamical system to that surface becomes zero, there should be a switch back to the original hybrid system. These observations allow one to formally complete a Lagrangian hybrid system by adding an additional *post-Zeno* domain of constrained motion to the system [2, 18].

In this paper, we study periodic orbits for completed Lagrangian hybrid systems, that pass through both the original and the *post-Zeno* domains of the hybrid system. Such periodic orbits are of paramount importance to a wide variety of applications, e.g., this is the type of orbits one obtains in bipedal robots. In particular, we begin by considering a *simple* periodic orbit which is an orbit that contains a single event of *perfectly plastic impact*. That is, after the impact, the system instantly switches to the post-Zeno domain. The key question is: *what happens to a simple periodic orbit when the impacts are not perfectly plastic*? The main result of this paper guarantees existence of a periodic orbit for completed Lagrangian hybrid system with non-plastic impacts given a stable periodic for the same system with plastic impacts; moreover, we give explicit bounds on the degree of plasticity that ensures the existence of such orbit.

The importance of the main result of this paper lies in the fact that impacts in mechanical systems are *never perfectly plastic*, so it is important to understand what happens to periodic orbits for perfectly plastic impacts in the case of non-plasticity. Using the example of a bipedal robot with knees [8, 13, 22], the knee locking (leg straightening) is modeled as a perfectly plastic impact. If one were to find a walking gait for this biped under this assumption, the main result of this paper would ensure that there would also be a walking gait in the case when the knee locking is not perfectly plastic, as would be true in reality. In light of this example, we conclude the paper by applying the main result of this paper to a double pendulum with a mechanical stop, which models a single leg of a bipedal robot with knees.

Both periodic orbits and Zeno behavior have been well-studied in the literature although they have yet to be studied simultaneously. With regard to Zeno behavior, it has been studied in the context of mechanical systems in [14, 17] with results that complement the results of this paper, and studied for other hybrid models in [6, 10, 21, 23, 27]. Periodic orbits have primarily been studied in hybrid systems in the context of bipedal locomotion for dynamic walking [8, 9, 15] and running [7], assuming perfectly plastic impacts. The pioneering work in [4] focuses on design of stable tracking control for cyclic tasks with Zeno behavior in Lagrangian hybrid systems, assuming that the system is *fully actuated*, i.e., all degrees-of-freedom are controlled. Note, however, that this assumption generally does not hold for locomotion systems, which are essentially *underactuated*.

2 Lagrangian Hybrid Systems

In this section, we introduce the notion of a hybrid Lagrangian and the associated Lagrangian hybrid system. Hybrid Lagrangians of this form have been studied in the context of Zeno behavior and reduction; see [1] and [17]. We begin this section by reviewing the notion of a simple hybrid system.

Definition 1. *A simple hybrid system is a tuple* $\mathscr{H} = (D, G, R, f)$*, where*

- D *is a smooth manifold called the domain,*
- G *is an embedded submanifold of* D *called the guard,*
- R *is a smooth map* $R : G \to D$ *called the reset map,*
- f *is a vector field on the manifold* D*.*

Hybrid executions. A *hybrid execution* of a simple hybrid system $\mathscr{H}$ is a tuple $\chi = (\Lambda, \mathcal{I}, \mathcal{C})$, where

- $\Lambda = \{0, 1, 2, \ldots\} \subseteq \mathbb{N}$ is an indexing set.
- $\mathcal{I} = \{I_i\}_{i\in\Lambda}$ is a *hybrid interval* where $I_i = [t_i, t_{i+1}]$ if $i, i+1 \in \Lambda$ and $I_{N-1} = [t_{N-1}, t_N]$ or $[t_{N-1}, t_N)$ or $[t_{N-1}, \infty)$ if $|\Lambda| = N$, N finite. Here, $t_i, t_{i+1}, t_N \in \mathbb{R}$ and $t_i \leq t_{i+1}$.
- $\mathcal{C} = \{c_i\}_{i\in\Lambda}$ is a collection of integral curves of f, i.e., $\dot{c}_i(t) = f(c_i(t))$ for $t \in I_i$, $i \in \Lambda$,

And the following conditions hold for every $i, i+1 \in \Lambda$:

(i) $c_i(t_{i+1}) \in G$,
(ii) $R(c_i(t_{i+1})) = c_{i+1}(t_{i+1})$,
(iii) $t_{i+1} = \min\{t \in I_i : c_i(t) \in G\}$.

The *initial condition* for the hybrid execution is $c_0(t_0)$.

Lagrangians. Let $q \in \mathbb{R}^n$ be the *configuration* of a mechanical system[3]. In this paper, we will consider Lagrangians, $L : \mathbb{R}^{2n} \to \mathbb{R}$, describing mechanical, or robotic, systems, which are Lagrangians of the form $L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q)$,

[3] For simplicity, we assume that the configuration space is identical to $\mathbb{R}^n$

where $M(q)$ is the (positive definite) inertial matrix, $\frac{1}{2}\dot{q}^T M(q)\dot{q}$ is the kinetic energy and $V(q)$ is the potential energy. We will also consider a *control law* $u(q,\dot{q})$, which is a given smooth function $u : \mathbb{R}^{2n} \to \mathbb{R}^n$. In this case, the Euler-Lagrange equations yield the (unconstrained, controlled) equations of motion for the system:

$$M(q)\ddot{q} + C(q,\dot{q}) + N(q) = u(q,\dot{q}), \tag{1}$$

where $C(q,\dot{q})$ is the vector of centripetal and Coriolis terms (cf. [16]) and $N(q) = \frac{\partial V}{\partial q}(q)$. Defining the *state* of the system as $x = (q,\dot{q})$, the Lagrangian vector field, f_L, associated to L takes the familiar form:

$$\dot{x} = f_L(x) = \begin{pmatrix} \dot{q} \\ M(q)^{-1}(-C(q,\dot{q}) - N(q) + u(q,\dot{q})) \end{pmatrix}. \tag{2}$$

This process of associating a dynamical system to a Lagrangian will be mirrored in the setting of hybrid systems. First, we introduce the notion of a hybrid Lagrangian.

Definition 2. *A simple hybrid Lagrangian is defined to be a tuple* $\mathbf{L} = (Q, L, h)$, *where*

- Q *is the configuration space (assumed to be identical to* $\mathbb{R}^n$*),*
- $L : TQ \to \mathbb{R}$ *is a hyperregular Lagrangian,*
- $h : Q \to \mathbb{R}$ *provides a unilateral constraint on the configuration space; we assume that the zero level set* $h^{-1}(0)$ *is a smooth manifold.*

Simple Lagrangian hybrid systems. For a given Lagrangian, there is an associated dynamical system. Similarly, given a hybrid Lagrangian $\mathbf{L} = (Q, L, h)$ the *simple Lagrangian hybrid system* associated to $\mathbf{L}$ is the simple hybrid system $\mathscr{H}_{\mathbf{L}} = (D_{\mathbf{L}}, G_{\mathbf{L}}, R_{\mathbf{L}}, f_{\mathbf{L}})$. First, we define

$$\begin{aligned} D_{\mathbf{L}} &= \{(q,\dot{q}) \in TQ : h(q) \geq 0\}, \\ G_{\mathbf{L}} &= \{(q,\dot{q}) \in TQ : h(q) = 0 \text{ and } dh(q)\dot{q} \leq 0\}, \end{aligned}$$

where $dh(q) = [\frac{\partial h}{\partial q}(q)]^T = [\, \frac{\partial h}{\partial q_1}(q) \; \cdots \; \frac{\partial h}{\partial q_n}(q)\,]$. In this paper, we adopt the reset map ([5]) $R_{\mathbf{L}}(q,\dot{q}) = (q, P_{\mathbf{L}}(q,\dot{q}))$, which is based on the *impact equation*

$$P_{\mathbf{L}}(q,\dot{q}) = \dot{q} - (1+e)\frac{dh(q)\dot{q}}{dh(q)M(q)^{-1}dh(q)^T} M(q)^{-1} dh(q)^T, \tag{3}$$

where $0 \leq e \leq 1$ is the *coefficient of restitution*, which is a measure of the energy dissipated through impact. This reset map corresponds to rigid-body collision under the assumption of *frictionless impact*. Examples of more complicated collision laws that account for friction can be found in [5] and [24]. Finally, $f_{\mathbf{L}} = f_L$ is the Lagrangian vector field associated to L in (2).

3 Zeno Behavior and Completed Hybrid Systems

In this section we define Zeno behavior in Lagrangian hybrid systems, introduce the notion of a completed hybrid system ([2, 18]), and define the notions of simple periodic orbit and Zeno periodic orbit, corresponding to periodic completed executions under plastic and non-plastic impacts. Then we define the stability of periodic orbits.

Zeno behavior. A hybrid execution χ is *Zeno* if $\Lambda = \mathbb{N}$ and $\lim_{i\to\infty} t_i = t_\infty < \infty$. Here t_∞ is called the *Zeno time*. If χ is a Zeno execution of a Lagrangian hybrid system $\mathscr{H}_{\mathbf{L}}$, then its *Zeno point* is defined to be

$$x_\infty = (q_\infty, \dot{q}_\infty) = \lim_{i\to\infty} c_i(t_i) = \lim_{i\to\infty} (q_i(t_i), \dot{q}_i(t_i)).$$

These limit points essentially lie on the *constraint surface* in state space, which is defined by $\mathcal{S} = \{(q, \dot{q}) \in I\!R^{2n} : h(q) = 0 \text{ and } dh(q)\dot{q} = 0\}$.

Constrained dynamical systems. We now define the holonomically constrained dynamical system $\mathscr{D}_{\mathbf{L}}$ associated with the hybrid Lagrangian $\mathbf{L}$. For such systems, the constrained equations of motion can be obtained from the equations of motion for the unconstrained system (1), and are given by (cf. [16])

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + N(q) = dh(q)^T\lambda + u(q, \dot{q}), \tag{4}$$

where λ is the Lagrange multiplier which represents the contact force. Differentiating the constraint equation $h(q) = 0$ twice with respect to time and substituting the solution for $\ddot{q}$ in (4), the solution for the constraint force λ is obtained as follows:

$$\begin{aligned} \lambda(q, \dot{q}) = & \left(dh(q)M(q)^{-1}dh(q)^T\right)^{-1} \\ & \left(dh(q)M(q)^{-1}(C(q, \dot{q})\dot{q} + N(q) - u(q, \dot{q})) - \dot{q}^T H(q)\dot{q}\right). \end{aligned} \tag{5}$$

From the constrained equations of motion, for $x = (q, \dot{q})$, we get the vector field

$$\dot{x} = \tilde{f}_L(x) = \begin{pmatrix} \dot{q} \\ M(q)^{-1}(-C(q, \dot{q})\dot{q} - N(q) + u(q, \dot{q}) + dh(q)^T\lambda(q, \dot{q})) \end{pmatrix}$$

Note that $\tilde{f}_L$ defines a vector field on the manifold $TQ|_{h^{-1}(0)}$, from which we obtain the dynamical system $\mathscr{D}_{\mathbf{L}} = (TQ|_{h^{-1}(0)}, \tilde{f}_L)$. For this dynamical system, $q(t)$ slides along the constraint surface $\mathcal{S}$ as long as the constraint force λ is positive.

A *constrained execution* $\tilde{\chi}$ of $\mathscr{D}_{\mathbf{L}}$ is a pair $(\tilde{I}, \tilde{c})$ where $\tilde{I} = [\tilde{t}_0, \tilde{t}_f] \subset I\!R$ if $\tilde{t}_f$ is finite and $\tilde{I} = [\tilde{t}_0, \tilde{t}_f) \subset I\!R$ if $\tilde{t}_f = \infty$, and $\tilde{c} : \tilde{I} \to TQ$, with $\tilde{c}(t) = (q(t), \dot{q}(t))$ a solution to the dynamical system $\mathscr{D}_{\mathbf{L}}$ satisfying the following properties:

$$\begin{array}{ll} \text{(i)} & h(q_0(\tilde{t}_0)) = 0, \\ \text{(ii)} & dh(q_0(\tilde{t}_0))\dot{q}_0(\tilde{t}_0) = 0, \\ \text{(iii)} & \lambda(q(\tilde{t}_0), \dot{q}(\tilde{t}_0)) > 0, \\ \text{(iv)} & \tilde{t}_f = \min\{t \in \tilde{I} : \lambda(q(t), \dot{q}(t)) = 0\}. \end{array} \tag{6}$$

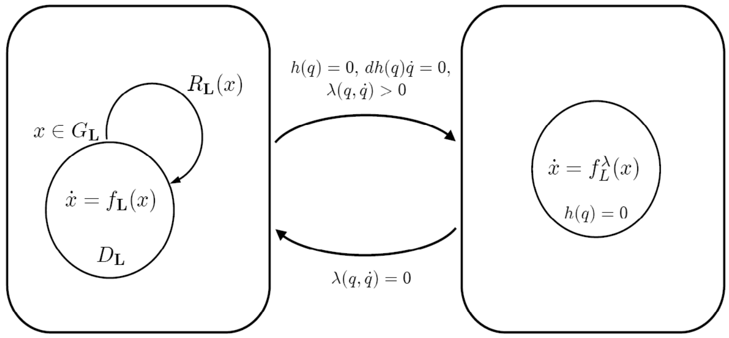

Fig. 1. A graphical representation of a completed hybrid system.

Using the notation and concepts introduced thus far, we introduce the notion of a completed hybrid system.

Definition 3. *If* $\mathbf{L}$ *is a simple hybrid Lagrangian and* $\mathscr{H}_{\mathbf{L}}$ *the corresponding Lagrangian hybrid system, the corresponding completed Lagrangian hybrid system*[4] *is defined to be:*

$$\overline{\mathscr{H}}_{\mathbf{L}} := \begin{cases} \mathscr{D}_{\mathbf{L}} & \text{if} \quad h(q) = 0\ ,\ dh(q)\dot{q} = 0, \text{ and } \lambda(q,\dot{q}) > 0 \\ \mathscr{H}_{\mathbf{L}} & \text{otherwise.} \end{cases}$$

Remarks. The system $\overline{\mathscr{H}}_{\mathbf{L}}$ can be viewed simply as a hybrid system with two domains; in this case, the reset maps are the identity, and the guards are given as in Fig. 1. Also note that the only way for the transition to be made from the hybrid system $\mathscr{H}_{\mathbf{L}}$ to the constrained system $\mathscr{D}_{\mathbf{L}}$ is if a specific Zeno execution reaches its Zeno point. Second, a transition for $\mathscr{D}_{\mathbf{L}}$ to $\mathscr{H}_{\mathbf{L}}$ happens when the constraint force λ crosses zero. Finally, it is shown in [18] that the constraint acceleration $\ddot{h}(q,\dot{q})$ and the constraint force $\lambda(q,\dot{q})$ in (5) satisfy *complementarity relation*. That is, while sliding along the constraint surface $\mathcal{S}$, either $\ddot{h} = 0$ and $\lambda > 0$, corresponding to maintaining constrained motion, or $\ddot{h} > 0$ and $\lambda = 0$, corresponding to leaving the constraint surface and switching back to the hybrid system. Thus, the definition of the completed hybrid system is consistent.

The completed execution. Having introduced the notion of a completed hybrid system, we must introduce the semantics of solutions of systems of this form. That is, we must introduce the notion of a *completed execution*.

Definition 4. *Given a simple hybrid Lagrangian* $\mathbf{L}$ *and the associated completed system* $\overline{\mathscr{H}}_{\mathbf{L}}$*, a* completed execution $\overline{\chi}$ *is a (possibly infinite) ordered sequence of alternating constrained and hybrid executions* $\overline{\chi} = \{\tilde{\chi}^{(1)},\ \chi^{(2)},\ \tilde{\chi}^{(3)},\ \chi^{(4)},\ \ldots\}$,

[4] As was orginally pointed out in [2], this terminology (and notation) is borrowed from topology, where a metric space can be completed to ensure that "limits exist."

with $\tilde{\chi}^{(i)}$ *and* $\chi^{(j)}$ *executions of* $\mathscr{D}_{\mathbf{L}}$ *and* $\mathscr{H}_{\mathbf{L}}$*, respectively, that satisfy the following conditions:*

(i) For each pair $\tilde{\chi}^{(i)}$ *and* $\chi^{(i+1)}$*,*

$$\tilde{t}_f^{(i)} = t_0^{(i+1)} \text{ and } \tilde{c}^{(i)}(\tilde{t}_f^{(i)}) = c_0^{(i+1)}(t_0^{(i+1)}).$$

(ii) For each pair $\chi^{(i)}$ *and* $\tilde{\chi}^{(i+1)}$*,*

$$t_\infty^{(i)} = \tilde{t}_0^{(i+1)} \text{ and } c_\infty^{(i)} = \tilde{c}^{(i+1)}(\tilde{t}_0^{(i+1)}).$$

where the superscript (i) *denotes values corresponding to the* i^{th} *execution in* $\overline{\chi}$*, and* $t_\infty^{(i)}, c_\infty^{(i)}$ *denote the Zeno time and Zeno point associated with the* i^{th} *hybrid execution* $\chi^{(i)}$*.*

Periodic orbits of completed hybrid systems. In the special case of plastic impacts $e = 0$, a *simple periodic orbit* is a completed execution $\overline{\chi}$ with initial condition $\tilde{c}^{(1)}(0) = x^*$ that satisfies $\tilde{c}^{(3)}(\tilde{t}_0^{(3)}) = x^*$. The period of $\overline{\chi}$ is $T = \tilde{t}_0^{(3)}$. In other words, this orbit consists of a constrained execution starting at x^*, followed by a hybrid (unconstrained) execution which is ended by a single plastic collision at $t = T$, that resets the state back to x^*.

For non-plastic impacts $e > 0$, a *Zeno periodic orbit* is a completed execution $\overline{\chi}$ with initial condition $\tilde{c}^{(1)}(0) = x^*$ that satisfies $c_\infty^{(2)} = \tilde{c}^{(3)}(\tilde{t}_0^{(3)}) = x^*$. The period of $\overline{\chi}$ is $T = t_\infty^{(2)} = \tilde{t}_0^{(3)}$. In other words, this orbit consists of a constrained execution starting at x^*, followed by a Zeno execution with infinite number of non-plastic impacts, which converges in finite time back to x^*.

Stability of hybrid periodic orbits. We now define the stability of hybrid periodic orbits.

Definition 5. *A Zeno (or simple) periodic orbit* $\overline{\chi} = \{\tilde{\chi}^{(1)}, \chi^{(2)}, \tilde{\chi}^{(3)}, \chi^{(4)}, ...\}$ *with initial condition* $x^* \in \mathcal{S}$ *is* locally exponentially stable *if there exist a neighborhood* $U \subset \mathcal{S}$ *of* x^* *and a scalar* $\gamma \in (0,1)$ *such that for any initial condition* $x_0 = \tilde{c}^{(1)}(0) \in U$*, the resulting completed execution satisfies* $\|\tilde{c}^{(2k+1)}(\tilde{t}_0^{(2k+1)}) - x^*\| \leq \|x_0 - x^*\|\gamma^k$ *for* $k = 1, 2, \ldots$*.*

Choice of coordinates. In the rest of this paper, we assume that the generalized coordinates contain the constraint function h as a coordinate, i.e. $q = (z, h)$. This assumption is quite general, since a transformation to such coordinate set must exist, at least locally, due to the regularity of $h(q)$. The state of the system thus takes the form $x = (z, h, \dot{z}, \dot{h}) \in I\!R^{2n}$. When the coordinates take this special form, the reset map (3) simplifies to

$$P_{\mathbf{L}}(q, \dot{q}) = \begin{pmatrix} \dot{z} - (1+e)\dot{h}\eta(z) \\ -e\dot{h} \end{pmatrix}, \text{ where } \eta(z) = \left.\frac{[M^{-1}(q)]_{1\ldots n-1,n}}{[M^{-1}(q)]_{n,n}}\right|_{h=0}. \quad (7)$$

The instantaneous solution for the accelerations $\ddot{q}$ in (1) is given by

$$\ddot{q}(q, \dot{q}) = (\ddot{z}(q, \dot{q}), \ddot{h}(q, \dot{q})) = M(q)^{-1}\left(u(q, \dot{q}) - C(q, \dot{q}) - N(q)\right). \quad (8)$$

4 Main Result

In this section we present the main result of this paper, namely, conditions under which the existence and stability of a simple periodic orbit imply existence of a Zeno periodic orbit.

4.1 Statement of Main Result

Before stating this result, some preliminary setup is needed. We can write $x^* = (z^*, 0, \dot{z}^*, 0)$, and define three types of neighborhoods of x^* in three different subspaces of $I\!R^{2n}$. For $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 > 0$, the neighborhoods $\Omega_1(\epsilon_1)$, $\Omega_2(\epsilon_1, \epsilon_2)$ and $\Omega_4(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ are defined as follows.

$$\begin{aligned}
\Omega_1(\epsilon_1) &= \{(q, \dot{q}) : \ \ h = 0,\ \dot{h} = 0,\ \text{and } \|z - z^*\| < \epsilon_1\} \\
\Omega_2(\epsilon_1, \epsilon_2) &= \{(q, \dot{q}) : \ \ h = 0,\ \dot{h} = 0,\ \|z - z^*\| < \epsilon_1, \text{ and } \|\dot{z} - \dot{z}^*\| < \epsilon_2\} \\
\Omega_4(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) &= \{(q, \dot{q}) : \ \ \|z - z^*\| < \epsilon_1,\ \|\dot{z} - \dot{z}^*\| < \epsilon_2,\ 0 < h < \epsilon_3, \text{ and } |\dot{h}| < \epsilon_4\}
\end{aligned}$$

Assume we are given a control law $u(q, \dot{q})$ and a starting point $x^* \in \mathcal{S}$ for which there exists of a simple periodic periodic orbit $\overline{\chi}^*$ starting at x^* which is locally exponentially stable. Define $v^* = \left|\dot{h}_0^{(2)}(t_1^{(2)})\right|$, which is the pre-collision velocity at the single (plastic) collision in the periodic orbit. The following assumption is a direct implication of the stability of $\overline{\chi}^*$:

Assumption 1. *Assume that there exist $\epsilon_1, \epsilon_2 > 0$,$\kappa \geq 1$ and $\gamma \in (0, 1)$, such that for any initial condition $x_0 \in \Omega_2(\epsilon_1, \epsilon_2)$, the corresponding completed execution with $e = 0$ satisfies the two following requirements:*

$$\begin{aligned}
&(a) \qquad \tilde{c}^{(3)}(\tilde{t}_0^{(3)}) \in \Omega_2(\gamma\epsilon_1, \gamma\epsilon_2) \\
&(b) \qquad \left|\dot{h}_0(t_1^{(2)})\right| < \kappa v^*.
\end{aligned} \tag{9}$$

Setup. To provide the conditions needed for the main result, for the given ϵ_1, ϵ_2 and κ, let the neighborhood Ω be defined as $\Omega = \Omega_4(\epsilon_1, \epsilon_2, \epsilon_3, \kappa v^*)$ for some $\epsilon_3 > 0$, and define the following scalars:

$$\begin{aligned}
a_{min} &= -\max\nolimits_{(q,\dot{q})\in\Omega} \ddot{h}(q, \dot{q}) \\
a_{max} &= -\min\nolimits_{(q,\dot{q})\in\Omega} \ddot{h}(q, \dot{q}) \\
\delta &= \sqrt{\left|\tfrac{a_{max}}{a_{min}}\right|} \\
\dot{z}_{max} &= \|\dot{z}^*\| + \epsilon_2 \\
\ddot{z}_{max} &= \max\nolimits_{(q,\dot{q})\in\Omega} \|\ddot{z}(q, \dot{q})\| \\
\eta_{max} &= \max\nolimits_{z\in\Omega_1(\epsilon_1)} \|\eta(z)\|.
\end{aligned} \tag{10}$$

The following theorem establishes sufficient conditions for existence of a Zeno periodic orbit given a simple periodic orbit.

Theorem 1. *Consider a simple periodic orbit* $\overline{\chi}^*$ *which is locally exponentially stable, and the given* $\epsilon_1, \epsilon_2 > 0$*,* $\kappa \geq 1$*, and* $\gamma \in (0,1)$ *that satisfy Assumption 1. Then for a given coefficient of restitution* e*, if the neighborhood* Ω *and its associated scalars defined in (10) satisfy the following conditions:*

$$a_{max} \geq a_{min} > 0 \tag{11}$$

$$e\delta < 1 \tag{12}$$

$$\frac{2e\kappa v^*}{a_{min}(1-\delta e)} \dot{z}_{max} \leq \epsilon_1(1-\gamma) \tag{13}$$

$$\left(\frac{1+\delta}{1-\delta e} \eta_{max} + \frac{2}{a_{min}(1-\delta e)} \ddot{z}_{max} \right) e\kappa v^* \leq \epsilon_2(1-\gamma), \tag{14}$$

$$\frac{(e\kappa v^*)^2}{2a_{min}} \leq \epsilon_3 \tag{15}$$

then there exists a Zeno periodic orbit with initial condition within $\Omega_2(\epsilon_1, \epsilon_2)$*.*

4.2 Proof of the Main Result

Before proving Theorem 1, we must define some preliminary notation. Consider the completed execution $\overline{\chi}$ with $e = 0$, and the execution $\overline{\chi}'$ with $e > 0$ under the same given initial condition $x_0 \in \Omega_2(\epsilon_1, \epsilon_2)$. Since we are only interested in the first hybrid and constrained elements of $\overline{\chi}$ and $\overline{\chi}'$, we simplify the notation by defining $\overline{\chi} = \{\tilde{\chi},\ \chi, ...\}$ and $\overline{\chi}' = \{\tilde{\chi}',\ \chi', ...\}$. Since the constrained motion does not contain any collisions, it is clear that $\tilde{\chi} = \tilde{\chi}'$. Moreover, the hybrid executions χ and χ' are also identical until the first collision time, that is $c_0(t) = c_0'(t)$ for $t \in [t_0, t_1]$ and $t_1 = t_1'$. Therefore, we will compare the solutions $c_i'(t)$ and $c_i(t)$ for $i > 1$, i.e. after the time t_1.

We now give the outline of the proof, which is divided into three steps. The first step proves that if the hybrid execution χ' stays within the neighborhood Ω, then conditions (11) and (12) imply that it is a Zeno execution. Step 2 verifies that under conditions (13) and (14), the execution χ' actually stays within Ω. The results of these two steps are stated as two lemmas, whose detailed proofs are relegated to [19] due to space constraints. Finally, the third step utilizes the two previous steps to complete the proof of Theorem 1.

Step 1. Consider a neighborhood Ω that satisfies conditions (11) and (12), and assume that the trajectory of the hybrid execution χ' satisfies $c_i'(t) \in \Omega$ for all $t \in I_i',\ i \in \Lambda' \setminus \{0\}$. This assumption implies that the h-component of $c_i'(t) = (z_i'(t), h_i'(t), \dot{z}_i'(t), \dot{h}_i'(t))$ satisfies the second-order differential inclusion

$$\ddot{h}_i'(t) \in [-a_{max}, -a_{min}], \tag{16}$$

for all $t \in I_i',\ i \in \Lambda' \setminus \{0\}$. At each collision time $t_i',\ i \in \Lambda' \setminus \{0\}$, (16) is re-initialized according to the collision law (7) as

$$\dot{h}_{i+1}'(t_i') = -e\dot{h}_i'(t_i'), \text{ and } h_{i+1}'(t_i') = h_i'(t_i') = 0. \tag{17}$$

Let $\tau_i = t'_{i+1} - t'_i$, which is the time difference between consecutive collisions, and let $v_i = -\dot{h}'_{i-1}(t'_i)$, which is the pre-collision velocity at time t'_i. The following lemma summarizes results on the hybrid execution χ' under the differential inclusion (16).

Lemma 1 ([19]). *Assume that the hybrid execution χ' satisfies the differential inclusion (16) for all $t \in I'_i$, $i \in \Lambda' \setminus \{0\}$, and that a_{min}, a_{max}, and δ satisfy conditions (11) and (12). Then χ' is a Zeno execution with a Zeno time t_∞. Moreover, the solution $c'_i(t)$ satisfies the following for all $i \geq 1$*

$$v_i \leq v_1 (e\delta)^{i-1} \tag{18}$$

$$\left|\dot{h}'_i(t)\right| \leq v_1 \text{ for all } t \in I'_i \tag{19}$$

$$\tau_i \leq \frac{2ev_1}{a_{min}} (e\delta)^{i-1} \tag{20}$$

$$t'_\infty - t'_1 \leq \frac{2ev_1}{a_{min}(1-e\delta)} \tag{21}$$

$$h'_i(t) \leq \frac{e^2 v_1^2}{2a_{min}} \text{ for all } t \in I'_i. \tag{22}$$

The key idea in the proof is utilization of optimal control theory to find the "most unstable" execution under the differential inclusion (16) and the impact law (17). It is shown in [19] that all possible executions satisfy the bound $v_{i+1} \leq e\delta v_i$. Therefore, condition (12) implies that the v_i-s are bounded by the decaying geometric series (18). All other bounds in (19)-(22) are then implied by (18).

Step 2: We now verify that for any initial condition in $\Omega_2(\epsilon_1, \epsilon_2)$, the solution actually stays within Ω, as summarized in the following lemma.

Lemma 2 ([19]). *Consider a neighborhood Ω that satisfies conditions (11)-(14). Then for any initial condition $x_0 \in \Omega_2(\epsilon_1, \epsilon_2)$, the hybrid execution χ' is a Zeno execution that satisfies $c'_i(t) \in \Omega$ for all $t \in I'_i$, $i \in \Lambda' \setminus \{0\}$.*

The main idea of the proof in [19], is to assume that the execution initially stays within the neighborhood Ω, and use (18)-(22) to find bounds on $q(t), \dot{q}(t)$ during the execution. Then, conditions (11)-(15) guarantee that the execution does not leave Ω at all times.

Step 3: We now utilize Lemma 1 and Lemma 2 to prove the main result.

Proof (of Theorem 1). Consider the completed execution $\overline{\chi}' = \{\tilde{\chi}', \chi', ...\}$ with $e > 0$, under initial condition $x_0 \in \Omega_2(\epsilon_1, \epsilon_2)$. Lemma 1 and Lemma 2 imply that χ' is a Zeno execution which reaches $\mathcal{S}$ in time t_∞, and that $c'_i(t) \in \Omega$ for all $i \geq 1$. Define the function $\Phi :\ \Omega_2(\epsilon_1, \epsilon_2) \to \mathcal{S}$ as $\Phi(x_0) = c'_\infty$, under initial condition $c'_0(0) = x_0$. Note that Φ is well-defined, since for any initial condition

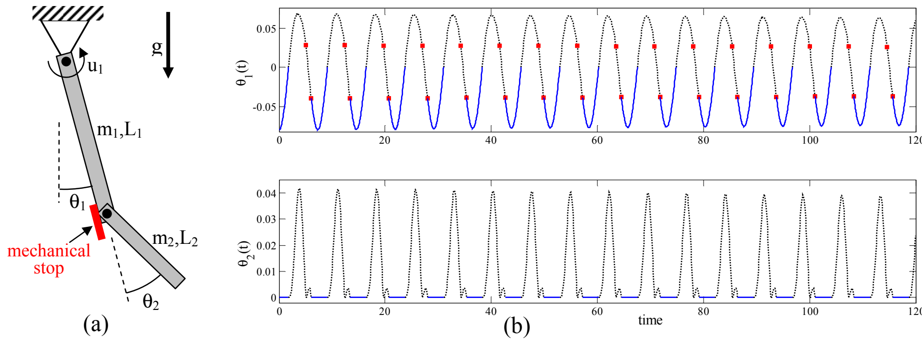

Fig. 2. (a) The constrained double pendulum system (b) Time plots of the solution $\theta_1(t)$ and $\theta_2(t)$ of the double pendulum with no actuation under plastic collisions.

within $\Omega_2(\epsilon_1, \epsilon_2)$, a Zeno execution is guaranteed. Moreover, since the limit point satisfies $c'_\infty \in \Omega \cap \mathcal{S} = \Omega_2(\epsilon_1, \epsilon_2)$, Φ maps $\Omega_2(\epsilon_1, \epsilon_2)$ onto itself. The continuity of the hybrid flow with respect to its initial condition, which is a fundamental property of a completed hybrid system with a single constraint (cf. [5]) implies that Φ is continuous. Invoking the *fixed point theorem* (cf. [11]), we conclude that there exists a fixed point $\bar{x} \in \Omega_2(\epsilon_1, \epsilon_2)$ such that $\Phi(\bar{x}) = \bar{x}$. Finally, the definition of Φ then implies that $\bar{x}$ corresponds to the starting point of a Zeno periodic orbit with period $T' = t'_\infty$.

5 Simulation Example

This section demonstrates the theoretical results on a constrained double pendulum, which is depicted in Figure 2(a). The double pendulum consists of two rigid links of masses m_1, m_2, lengths L_1, L_2, and uniform mass distribution, which are attached by revolute joints, while a mechanical stop dictates the range of motion of the lower link. The upper joint is actuated by a torque u_1, while the lower joint is passive. This example serves as a simplified model of a leg with a passive knee and a mechanical stop.

The configuration of the double pendulum is $q = (\theta_1, \theta_2)$, and the constraint that represents the mechanical stop is given by $h(q) = \theta_2 \geq 0$. Note that in that case the coordinates are already in the form $q = (z, h)$, where $z = \theta_1$ and $h = \theta_2$. The Lagrangian of the system is given by $L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + (\frac{1}{2}m_1 L_1 + m_2 L_1) g \cos\theta_1 + \frac{1}{2}m_2 L_2 g \cos(\theta_1 + \theta_2)$, with the elements of the 2×2 inertia matrix $M(q)$ given by $M_{11} = m_1 L_1^2/3 + m_2(L_1^2 + L_2^2/3 + L_1 L_2 \cos\theta_2)$, $M_{12} = M_{21} = m_2(3L_1 L_2 \cos\theta_2 + 2L_2^2)/6$, $M_{22} = m_2 L_2^2/3$. The values of parameters for the simulations were chosen as $m_1 = m_2 = L_1 = L_2 = g = 1$.

The first running simulation shows the motion of the *uncontrolled system* i.e. $u_1 = 0$, under *plastic collisions*, i.e. $e = 0$. Fig. 2(b) shows the time plots of $\theta_1(t)$ and $\theta_2(t)$ under initial condition $q(0) = (-0.08, 0)$ and $\dot{q}(0) = (0, 0)$. The parts of unconstrained motion appear as dashed curves, and the parts of constrained

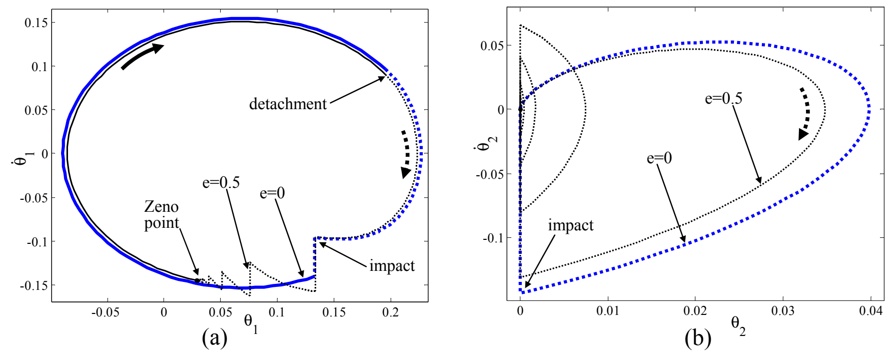

Fig. 3. Phase portraits of the periodic orbit in (a)$(\theta_1, \dot{\theta}_1)$-plane and (b)$(\theta_2, \dot{\theta}_2)$- plane for $e = 0$ (thin black) and $e = 0.5$ (thick blue)

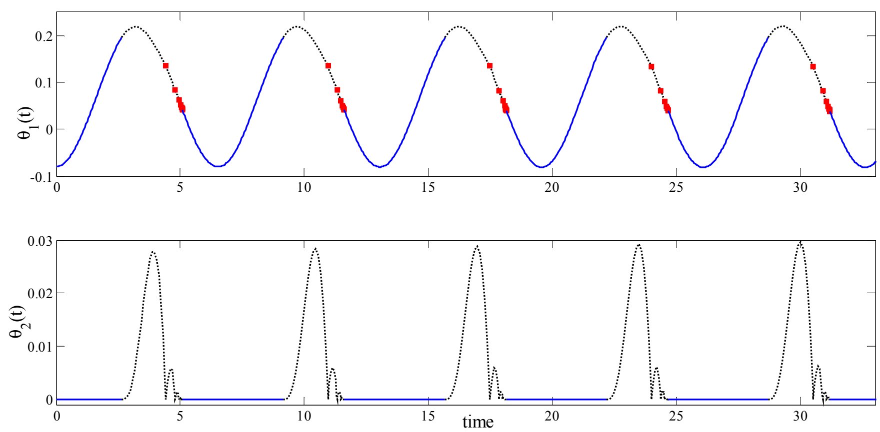

Fig. 4. Time plots of $\theta_1(t)$ and $\theta_2(t)$ for the controlled double pendulum with $e = 0.5$.

motion appear as solid curves. The points of collision events are marked with squares ('■') on the curve of $\theta_1(t)$. The double pendulum exhibits a slightly decaying periodic-like motion with *two* plastic collisions per cycle. At each cycle, after the first plastic collision, the constraint force λ required to maintain the constraint $\theta_2 = 0$ is negative. Thus, the lower link instantaneously detaches to another phase of unconstrained motion, until a second plastic collision occurs. After the second collision, the lower link locks at $\theta_2 = 0$, and the pendulum switches to a constrained motion with positive constraint force $\lambda > 0$ for some finite time, until λ crosses zero, and the lower link detaches again.

In order to obtain a non-decaying periodic solution with a *single* plastic collision per cycle, i.e., a simple periodic orbit, we add a PD control law for the torque u_1 as $u_1(\theta_1, \dot{\theta}_1) = -k_1(\theta_1 - \theta_{1e}) - c_1\dot{\theta}_1$. The control parameters are chosen as $k_1 = 0.5$, $\theta_{1e} = \pi/9$ and $c_1 = -0.01$. The proportional term associated with k_1 was chosen as to increase the positive acceleration $\ddot{\theta}_1$ and decrease the negative acceleration $\ddot{\theta}_2$ for $\theta_1 < 0$, and thus increase the constraint force λ that ensures that after the first collision, the lower link does not detach. The

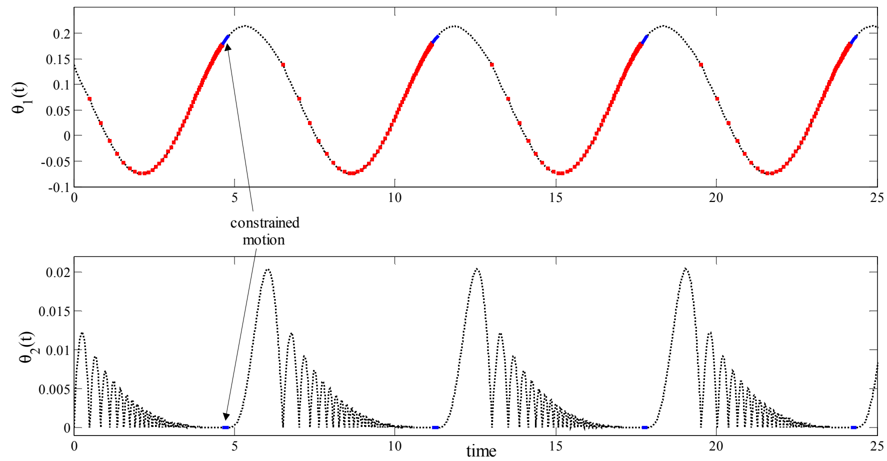

Fig. 5. Time plots of $\theta_1(t)$ and $\theta_2(t)$ for the controlled double pendulum with $e = 0.9225$.

negative dissipation term associated with c_1 injects a small amount of energy to the system, that compensates for the losses due to collisions. In simulation under the control law with the same initial condition as above, we obtained convergence to a simple periodic orbit with a single plastic collision per cycle. Figures 3(a) and 3(b) show the phase portraits of the periodic orbit in $(\theta_1, \dot{\theta}_1)$- and $(\theta_2, \dot{\theta}_2)$- planes, respectively. (Time plots of θ_1 and θ_2 appear in [19]).

Next, we apply Theorem 1 to check for existence of a Zeno periodic orbit with $e > 0$. One can verify numerically that the assumptions of the theorem are satisfied and that, in particular, the simple periodic orbit obtained through the control law is locally exponentially stable with $\gamma = 0.9404$. Choosing $\epsilon_1 = 0.017$, $\epsilon_2 = 0.06$ and $\epsilon_3 = 0.005$, Theorem 1 implies that the existence of a Zeno periodic orbit with initial condition within $\Omega_2(\epsilon_1, \epsilon_2)$ is guaranteed for any $e \leq 0.0015$. Simulation of the double-pendulum system with $e = 0.0015$ verifies the existence of a Zeno periodic orbit. The simulation results were not shown, since they are not visually distinguishable from the results with $e = 0$.

In order to illustrate the strong conservatism of Theorem 1, we conducted another simulation under the same initial condition, with a coefficient of restitution $e = 0.5$. The infinite Zeno executions were truncated after a finite number of collisions at which the collision velocity $\dot{h}$ is below a threshold of 10^{-10}. The simulation results, which are shown in the time plots of Figure 4, clearly indicate the existence of a Zeno periodic orbit, which was verified numerically to be also locally stable. Figures 3(a) and 3(b) show the phase portraits of the periodic orbits in $(\theta_1, \dot{\theta}_1)$- and $(\theta_2, \dot{\theta}_2)$- planes, respectively, for coefficients of restitution $e = 0$ (plastic impacts) and $e = 0.5$. The thick (blue) curves correspond to the case $e = 0$, and the thin (black) curves correspond to the case $e = 0.5$. The parts of unconstrained motion appear as dashed curves, and the parts of constrained motion appear as solid curves. Note that in Figure 3(b), the constrained motion collapses to the single point $(\theta_2, \dot{\theta}_2) = 0$. From the figures, one can clearly see how the simple periodic orbit is perturbed under non-plastic impacts.

Finally, we gradually increased the coefficient of restitution e and numerically checked for existence of Zeno periodic orbits. The largest value of e for which we obtained such an orbit was $e = 0.9225$. For this value of e, the duration of the constrained motion in the Zeno periodic orbit is very short, as shown in the simulation results of Figure 5. For larger values of e, *the phase of constrained motion vanishes*, and the execution is no longer Zeno. This transition can be viewed as a new type of bifurcation in Lagrangian hybrid systems, in which a Zeno periodic orbit ceases to be Zeno. To our knowledge, this type of bifurcation was never studied before in the recently emerging literature on bifurcations in non-smooth mechanical systems, (cf. [3, 12]).

6 Conclusion

This paper considered two types of periodic orbits in completed Lagrangian hybrid systems: *simple* and *Zeno*. The main result presented is sufficient conditions on when a simple periodic orbit in a Lagrangian hybrid system implies the existence of a Zeno periodic orbit in the same Lagrangian hybrid system with a different coefficient of restitution. Moreover, these conditions give an explicit upper bound the change in the coefficient of restitution that guarantees existence of the Zeno periodic orbit.

The results indicate two major future research directions: better bounds on the allowable change in the coefficient of restitution and conditions on the preservation of stability. For the first direction, as was illustrated by the example, the obtained bounds are strongly conservative; computing tighter bounds in a rigorous fashion will be practically useful and theoretically satisfying. The second future research direction—studying stability—is even more interesting. The authors have been able to show that under certain simplifying assumptions, stability of the simple periodic orbit directly implies the stability of the Zeno periodic orbit. However, this preliminary result was not included in the paper due to space constraints. In the future, understanding how stability extends from one type of orbit to the other with the fewest possible assumptions will provide new and interesting challenges. Finally, extending the results to Lagrangian hybrid system *with multiple constraints* will enable the analysis of full models of bipeds with knees for designing stable walking and running under non-plastic impacts.

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