Using the Set-Valued Bouncing Ball for Bounding Zeno Solutions of Lagrangian Hybrid Systems

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Abstract: In this paper, we study Zeno behavior in Lagrangian hybrid systems, which are mechanical systems with unilateral constraints that are undergoing impacts. Zeno solutions involve an infinite number of impacts occurring in a finite amount of time (the Zeno time). In such systems, one is typically not able to explicitly compute the Zeno time and Zeno limit point, and even not to detect a Zeno solution from its initial condition. We address these problems by replacing the nonlinear dynamics with a simple hybrid system whose dynamics is a set-valued version of the bouncing ball. We utilize optimal control analysis to derive conditions for the Zenoness of all solutions and compute bounds on their Zeno time and Zeno limit point, which also apply to solutions of the original Lagrangian hybrid system. Application of the results is demonstrated on a Lagrangian hybrid system with two degrees of freedom.

1. INTRODUCTION

Hybrid dynamical systems are systems that consist of both continuous-time and discrete-time dynamics (Goebel et al. [2009], van der Schaft and Schumacher [2000]). A fundamental phenomenon that is unique to hybrid systems is *Zeno behavior* (also known as *chattering*), where the solution involves an infinite number of discrete transitions occurring in finite time. Zeno behavior has recently gained increasing interest, in works studying conditions for its existence, (Heymann et al. [2005], Lamperski and Ames [2008]) and its relation to asymptotic stability (Goebel and Teel [2008b], Or and Ames [2010]). The classical example of Zeno behavior is the *bouncing ball* system, having one degree-of-freedom (DOF) which describes the motion of a rigid ball bouncing on a flat ground, where collisions of the ball with the ground are modeled as rigid-body impacts. A more complicated model which has been extensively investigated is that of a bouncing ball on a periodically vibrating table (Holmes [1982]), which, under suitable choice of parameters, also displays Zeno behavior (Luck and Mehta [1993], Heimsch and Leine [2009]).

The bouncing ball is only a special case of a more general class of systems — *Lagrangian hybrid systems*, which are mechanical systems with unilateral contacts that are undergoing impacts (Pfeiffer and Glocker [1996], Brogliato [1999]). In these systems, the configuration variables q must satisfy a unilateral constraint of the form $h(q) \geq 0$. Such systems often display Zeno behavior, where the solution converges in finite time to a limit point called *Zeno equilibrium*. Unlike the case of a bouncing ball, in such systems it is impossible to explicitly compute the Zeno time and Zeno equilibrium point. Moreover, even characterization of initial conditions that lead to Zeno solution is not a trivial task. Sufficient conditions for existence of Zeno behavior in such systems were recently derived in (Lamperski and Ames [2008]), where the main physical observation is that in a small neighborhood of a Zeno equilibrium, the dynamics of the constraint function $h(q(t))$ should be *similar to that of a bouncing ball.* However, no explicit bound on this neighborhood was given. Another related problem is derivation of practical bounds on Zeno point of a solution for purpose of numerical simulation. In (Nordmark and Piiroinen [2009]), an approximate map to the Zeno point is derived assuming sufficiently low impact velocity $\dot{h}$, but no explicit bounds are given.

A preliminary step towards addressing these problems was taken in our recent work (Or and Teel [2010]). The key idea was to replace the dynamics of the constraint function h by a *set-valued* version of the bouncing ball system. This defines a hybrid system called the *set-valued bouncing ball* (**SVBB**). Utilizing Lyapunov analysis and optimal control, we obtained conditions under which all possible solutions of the **SVBB** system are Zeno, and derived a tight bound on the Zeno time. In this paper, we generalize the **SVBB** model in order to account for the dynamics of another coordinate which is unconstrained, and introduce the hybrid system of the **SVBB2** — *Set-Valued Bouncing Ball with two degrees-of-freedom.* Using this system, we provide upper bounds on the drift in the unconstrained coordinate during a Zeno solution. We exploit the homogeneity (time-scaling) property of this system (Goebel and Teel [2008a], Schumacher [2009]) in order to solve an elementary optimization problem in continuous-time domain only, while avoiding the complication of optimal control theory for hybrid systems at its full generality (cf. Cassandras et al. [2001]). The results are applied to finding conditions for Zenoness of solutions of Lagrangian hybrid systems with two DOF and derivation of bounds on these solutions, as demonstrated in an example of a ball bouncing on a sinusoidal surface in two dimensions.

2. PRELIMINARIES AND PROBLEM STATEMENT

We begin by defining the general set-valued hybrid system and the Lagrangian hybrid system.

2.1 Hybrid systems

Let $F, G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be set-valued mappings and $C, D \subset \mathbb{R}^n$ be sets. We consider hybrid systems of the form

$$\mathcal{H}: \quad \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D. \end{cases} \tag{1}$$

For more background on hybrid systems in this framework, see (Goebel et al. [2009]). A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if $E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ ... \leq t_J$. It is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ...J\})$ is a compact hybrid time domain. Equivalently, E is a hybrid time domain if E is a union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, with the "last" interval possibly of the form $[t_j, T)$ with T finite or $T = +\infty$. A *hybrid arc* is a function ϕ whose domain dom ϕ is a hybrid time domain and such that for each $j \in \mathbb{N}$, $t \to \phi(t, j)$ is locally absolutely continuous on $I_j := \{t \,|\, (t, j) \in \text{dom } \phi\}$. A hybrid arc ϕ is *complete* if its domain, dom ϕ, is unbounded. A hybrid arc ϕ is a *solution to the hybrid system* $\mathcal{H}$ if $\phi(0, 0) \in C \cup D$ and

(i) any $j \in \mathbb{N}$ such that $\text{int}(I_j) \notin \emptyset$ satisfies $\phi(t, j) \in C$ and $\dot{\phi}(t, j) \in F(\phi(t, j))$ for almost all $t \in \text{int}(I_j)$;
(ii) for all $(t, j) \in$ dom ϕ such that $(t, j + 1) \in$ dom ϕ: $\phi(t, j) \in D$ and $\phi(t, j + 1) \in G(\phi(t, j))$.

A solution ϕ is *maximal* if there does not exist a solution ψ with dom $\phi \subset$ dom ψ, dom $\phi \neq$ dom ψ, $\phi(t, j) = \psi(t, j)$ for all $(t, j) \in$ dom ϕ. Complete solutions are maximal.

Zeno solutions: A hybrid arc ϕ is called *Zeno* if it is complete but $T(\phi) := \sup\{t \in \mathbb{R}_{\geq 0} \,|\, \exists j \text{ s.t. } (t, j) \in \text{dom } \phi\}$ is finite. In words, ϕ is Zeno if it experiences infinitely many jumps in finite (ordinary) time. The limit $T(\phi)$ of ordinary time is called the *Zeno time* of ϕ. Finally, for a given initial condition $x_0 \in C \cup D$, let $T_{max}(x_0)$ denote the supermum of $T(\phi)$ over all possible solutions ϕ satisfying $\phi(0, 0) = x_0$.

2.2 Lagrangian hybrid systems

Lagrangian hybrid systems are unilaterally constrained mechanical systems that are undergoing impacts. The configuration of the system is described by generalized coordinates $q \in \mathbb{R}^n$, and is bounded by a unilateral constraint $h(q) \geq 0$, which typically represents impenetrability between solid bodies. When the configuration satisfies $h(q) > 0$, the dynamics of the system is governed by Euler-Lagrange Equations, given by

$$M(q)\ddot{q} + B(q, \dot{q}) + \mathcal{G}(q) = 0, \tag{2}$$

where $M(q)$ is called the matrix of inertia, $B(q, \dot{q})$ describes centripetal and Coriolis forces, and $\mathcal{G}(q)$ represents potential forces such as gravity. When the solution $(q, \dot{q})$ of (2) hits the constraint, i.e. $h(q(t)) = 0$ and $\nabla h(q(t)) \cdot \dot{q}(t) < 0$, the system undergoes a collision which results in an impact event. This event is modelled as an instantaneous jump in the velocities $\dot{q}$ while the configuration q remains unchanged. The jump rule for the velocity, called the *impact law* is given by

$$\dot{q} \to P(q, \dot{q}), \text{ where } P(q, \dot{q}) = \dot{q} - (1 + e)\frac{\nabla h(q)\dot{q}}{\nabla h(q) M(q)^{-1} \nabla h(q)^T} M(q)^{-1} \nabla h(q)^T. \tag{3}$$

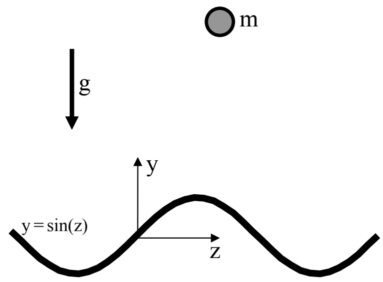

Fig. 1. The bouncing ball on a sinusoidal surface

It can be verified that the impact law (3) implies that the velocity of the constraint function $\dot{h} = \nabla h(q) \cdot \dot{q}$ is mapped to $-e\dot{h}$. The scalar $e \in (0, 1)$ is called the *coefficient of restitution.* The dynamics of Lagrangian hybrid systems can be easily cast into the hybrid system formulation (1) by defining the state vector $x = (x_1, x_2)$, where $x_1 = q$ and $x_2 = \dot{q}$. The discrete- and continuous-time dynamics are then given by

$$F(x) = \begin{bmatrix} x_2 \\ -M^{-1}(x_1)(B(x_1, x_2) + \mathcal{G}(x_1)) \end{bmatrix}, \ G(x) = \begin{bmatrix} x_1 \\ P(x_1, x_2) \end{bmatrix}.$$

The domains of the hybrid systems are given by $C = \{(x_1, x_2) : \ h(x_1) \geq 0\}$ and $D = \{(x_1, x_2) : \ h(x_1) = 0 \text{ and } \nabla h(x_1) \cdot x_2 < 0\}$. Note that this hybrid system is single valued rather than set valued, and thus has a unique solution under any given initial condition.

Choice of coordinates: We now define a convenient set of coordinates for a Lagrangian hybrid system. In this set, the constraint function h is chosen as the first coordinate, and $z \in \mathbb{R}^{n-1}$ are the unconstrained coordinates, so that $q = (h, z)$. This choice is quite general, since a transformation to such coordinate set must exist, at least locally, assuming that 0 is a regular value of $h(q)$.

Example - the BBSS system: As an example, we consider a Lagrangian hybrid system called **BBSS** — *Bouncing Ball on a Sinusoidal Surface.* The ball is modelled as a point mass m, and the generalized coordinates $q = (y, z)$ represent its position in the plane (Fig. 1). The sinusoidal surface imposes a unilateral constraint on the coordinates, given by $h(q) = y - \sin(z) \geq 0$. The dynamic equation governing the unconstrained motion of the ball for $h(q) > 0$ is of the form (2), where $M(q) = \text{diag}\{m, m\}$, $B(q, \dot{q}) = 0$ and $\mathcal{G}(q) = (0, mg)$, where g is the acceleration of gravity. Using the transformation of coordinates defined above, we choose the new set of coordinates as $q = (h, z)$. In this set of coordinates, the unconstrained equations of motion are given by

$$\ddot{h} = -g + \dot{z}^2 \sin(z) \text{ and } \ddot{z} = 0. \tag{4}$$

The impact law $\dot{q} \to P(q, \dot{q})$ in (3) is written in these coordinates as

$$\begin{pmatrix} \dot{h} \\ \dot{z} \end{pmatrix} \to \begin{pmatrix} -e\dot{h} \\ \dot{z} + (1 + e)\dot{h}\dfrac{\cos(z)}{1 + \cos(z)^2} \end{pmatrix}. \tag{5}$$

2.3 Problem Statement

Having defined all the terminology, our problem can now be stated as follows:

Given a Lagrangian hybrid system of the form (2), a constraint function $h(q) \geq 0$, and initial condition x_0,

(i) Find sufficient conditions guaranteeing that the solution under initial condition x_0 is Zeno.
(ii) Find bounds on the Zeno limit point and the Zeno time of the solution under initial condition x_0.

The key insight in addressing this problem is the observation that along a Zeno solution, the constraint function h behaves roughly like a bouncing ball, where the constant acceleration of gravity is replaced by a state-dependent term. This fact motivates the use of a set-valued hybrid system, as detailed below.

2.4 The **SVBB** *and* **SVBB2** *hybrid system*

The set-valued bouncing ball with two degrees-of-freedom (**SVBB2**) is a hybrid system with state $x \in \mathbb{R}^2$ and data

$$C = \left\{x \in \mathbb{R}^4 : x_1 \geq 0\right\}, \quad D = \left\{x \in \mathbb{R}^4 : x_1 = 0,\ x_2 \leq 0\right\},$$
$$F(x) = \left\{\begin{bmatrix} x_2 \\ -a \\ x_4 \\ b \end{bmatrix}, a \in [a_{min}, a_{max}]\right\}, \quad G(x) = \begin{bmatrix} 0 \\ -ex_2 \\ x_3 \\ x_4 - \gamma x_2 \end{bmatrix}, \tag{6}$$

where $e \in (0, 1)$ and $b, \gamma \geq 0$. The state variables x_1 and x_2 represent the constrained coordinate h and its velocity $\dot{h}$, respectively, where a is its set-valued acceleration. The scalars a_{min} and a_{max} represent bounds on $\ddot{h}$ when evaluated in a small neighborhood of a Zeno equilibrium point. The states x_3 and x_4 represent the unconstrained coordinate z. The scalar b represents the acceleration $\ddot{z}$, while γ represents the influence of an impact on the change in the velocity $\dot{z}$. The hybrid dynamics of the x_1- and x_2-components in (6) is independent of x_3 and x_4, and defines the **SVBB** system studied in (Or and Teel [2010]).

We now review two key results from (Or and Teel [2010]). The first result derives conditions for Zenoness of all possible solutions, as summarized in the following theorem.

Theorem 1. (Or and Teel [2010]). *Consider the hybrid system of the 2-DOF set valued bouncing ball given in (6). Then for any given initial condition $x_0 \in C \cup D$, all possible solutions are Zeno if and only if the following conditions hold:*

$$a_{max} \geq a_{min} > 0 \text{ and } e^2\alpha < 1, \text{ where } \alpha = \frac{a_{max}}{a_{min}}. \tag{7}$$

This theorem is proven in (Or and Teel [2010]) by using both Lyapunov analysis and optimal control theory. The main idea of the proof is the observation that for any given solution $\phi(t, j)$ of the **SVBB2** system in (6), the series of post-impact $\dot{h}$ velocities $v_j = x_2(t_j, j)$ satisfies the bound

$$v_{j+1} \leq e\sqrt{\alpha} v_j, \tag{8}$$

where α is defined in (7). That is, the series v_j is bounded by a geometric series with a multiplying factor of $e\sqrt{\alpha}$. This factor must be less than one in order for all possible solutions to be Zeno, which gives the condition in (7).

The second result gives a tight bound on the Zeno time for all possible solutions of the **SVBB2** system under initial conditions such that $x_1 = 0$, as summarized in the following theorem.

Theorem 2. (Or and Teel [2010]). *Consider all possible solutions $\phi(t, j)$ of the* **SVBB2** *system under initial condition $x_1(0, 0) = 0$ and $x_2(0, 0) = \nu \geq 0$. Assuming that condition (7) is satisfied, all solutions are Zeno, and their maximal Zeno time $T_{max}(x_0)$ is given by*

$$T_{max}(x_0) = 2\frac{1 + e}{1 - e^2\alpha} \cdot \frac{\nu}{a_{min}}. \tag{9}$$

Theorems 1 and 2 only involve the dynamics of x_1 and x_2, which dictate the Zenoness of a solution and its Zeno time.

3. OPTIMAL CONTROL ANALYSIS

In this section, we utilize optimal control analysis in order to derive bounds on the values of x_3 and x_4 at the Zeno limit point of the **SVBB2** system. The derivation is based on the classical notion of Pontryagin's maximum principle from optimal control theory (cf. Bryson and Ho [1975]).

3.1 Bound on the Zeno value of x_4 in the **SVBB2** *system*

Consider the **SVBB2** system (6), and assume that condition (7) is satisfied, so that all possible solutions are Zeno. Focusing on initial conditions such that $x_1 = 0$ and $x_2 > 0$, the following theorem gives a tight upper bound on the Zeno limit value of x_4, defined as $x_4^\infty = \lim_{j\to\infty} x_4(t_j, j)$.

Theorem 3. *Consider the* **SVBB2** *system in (6) under initial condition $\phi(0, 0) = (0, \nu, x_{30}, x_{40})^T$, where $\nu > 0$. Assuming that condition (7) is satisfied, all solutions are Zeno, and the maximal value of x_4^∞ over all possible solutions is given by*

$$\begin{aligned} &x_{4max} = x_{40} + \\ &\frac{\beta(\kappa(\alpha - 1) + 1) + (\beta + \alpha\gamma)\sqrt{(\kappa - 1)^2 + \alpha\kappa(2 - \kappa)}}{\alpha(1 - e\sqrt{(\kappa - 1)^2 + \kappa(2 - \kappa)})}\nu, \\ &\text{where } \kappa = \frac{\Delta - \beta e(\beta(e + 1) + \alpha\gamma) + \beta\sqrt{\Delta}}{\Delta + \beta^2 e^2(\alpha - 1)}, \quad \beta = \frac{b}{a_{min}}, \\ &\text{and} \quad \Delta = \beta^2(1 + e)^2 + 2\beta(\alpha e + 1)\gamma + \alpha\gamma^2. \end{aligned} \tag{10}$$

Proof: First, note that the changes in x_4 in (6) are only additive. That is, one only needs to maximize the difference $x_4^\infty - x_{40}$. Moreover, the changes in x_4 in (6) are independent of x_3, so that the bound on x_4^∞ depends on ν only. Let $\phi_\nu^*(t, j)$ denote the *optimal solution* of (6) that maximizes the difference $x_4^\infty - x_{40}$ under initial condition $x_0 = (0, \nu, x_{30}, x_{40})^T$. We now make two key observations, as follows. The first observation is that any "tail" of an optimal solution is also an optimal solution. Therefore, denoting v_k as the x_2-component of $\phi_\nu^*(t_k, k)$ for some $k \in I\!N$, one obtains $\phi_{v_k}^*(t, j) = \phi_\nu^*(t + t_k, j + k)$. The second observation is that the **SVBB2** system satisfies the property of *homogeneity* (cf. Goebel and Teel [2008a], see also Schumacher [2009]). In particular, it can be verified that for any $c > 0$, $\phi(t, j)$ is a solution of (6) if and only if $M(c) \cdot \phi(t/c, j)$ is also a solution of (6), where $M(c) = \mathsf{diag}(c^2, c, c^2, c)$. Therefore, for any $c > 0$ one gets $\phi_{c\nu}^*(t, j) = M(c) \cdot \phi_\nu^*(t/c, j)$. In words, if ω^* is the maximal value of $x_4^\infty - x_{40}$ under initial velocity ν which is attained by the solution $\phi_\nu^*(t,j)$, then scaling the initial velocity ν by c results in a maximal value of $c\omega^*$, which is attained by the scaled solution $M(c)\cdot\phi_\nu^*(t/c,j)$. Combining these two observations together implies the existence of a scalar $\eta \in (0,1)$ such that for any $(t,j) \in \text{dom } \phi_\nu^*$, one has

$$\phi_\nu^*(t,j) = M(\eta^j)\cdot\phi_\nu^*(\eta^{-j}(t-t_j),0). \tag{11}$$

That is, the behavior of the optimal solution in the jth interval of ordinary time is identical to its behavior in the first time interval up to scaling of magnitude and time. The factor η in (11) is related to the x_2-component of the optimal solution by $\eta = x_2^*(t_1,1)/\nu = -ex_2^*(t_1,0)/\nu$. Moreover, the x_4- component of the optimal solution satisfies $x_4^*(t,j) - x_4^*(t_j,j) = \eta^j(x_4^*(\eta^{-j}(t-t_j),0) - x_{40})$. Using these scaling relations and the dynamics of x_4 in (6), the total change in x_4 along the optimal solution is given by the geometric series:

$$\begin{aligned} x_{4max} - x_{40} &= \sum_{j=0}^{\infty} b(t_{j+1}-t_j) - \gamma x_2^*(t_{j+1}, j+1) \\ &= (bt_1 - \gamma x_2^*(t_1,1))\sum_{j=0}^{\infty}\eta^j = \frac{bt_1 - \gamma x_2^*(t_1,1)}{1-\eta}. \end{aligned} \tag{12}$$

In order to find the maximal value x_{4max}, one only needs to find the optimal value of the factor η. This reduces to solving a problem of optimal control on the first time interval only, as follows. Consider the control system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \text{where } u \in [-a_{max}, -a_{min}]. \tag{13}$$

This system represents the differential inclusion $\dot{x} \in F(x)$ for the (x_1,x_2)-components of $F(x)$ as given in (6). We view the initial and final times τ_0 and τ_f as the endpoints of the first time interval $[0,t_1]$ in a solution $\phi(t,j)$ of the **SVBB2** system. The initial condition of (13) is thus given by $x_1(\tau_0) = 0$, $x_2(\tau_0) = \nu > 0$. One end condition is specified, namely $x_1(\tau_f) = 0$. However, the end time τ_f, as well as $x_{2f} = x_2(\tau_f)$, which corresponds to the pre-impact velocity $\dot{h}$ at τ_f, are both unspecified. The cost function to be maximized is given by $P(x_f,\tau_f) = \dfrac{b\tau_f - \gamma x_{2f}}{1+ex_{2f}/\nu}$. This cost function is obtained by subsituting $\eta = -ex_{2f}/\nu$ in (12). That is, it corresponds to the maximal difference $x_{4max} - x_{40}$ over all solutions of the **SVBB2** system. Using standard terminology of optimal control, the Hamiltonian of the system is given by $H(x,\lambda,u,t) = \lambda_1 x_2 + \lambda_2 u$. The co-state dynamic equation $\dot{\lambda} = -\partial H/\partial x$ then gives $\dot{\lambda}_1 = 0$, $\dot{\lambda}_2 = -\lambda_1$, indicating that λ_1 is constant and $\lambda_2(t)$ is a linear function of t. Pontryagin's maximum principle states that the optimal control input $u^*(t)$ is the one that maximizes the Hamiltonian H at all times. Therefore, the value of $u^*(t)$ can only be either $-a_{min}$ or $-a_{max}$, depending on the sign of $\lambda_2(t)$. The end condition on λ_2 is given by $\lambda_2(\tau_f) = \dfrac{\partial P}{\partial x_{2f}} = -\nu\dfrac{\gamma\nu + e\tau_f}{(\nu+ex_{2f})^2} < 0$. Since $\lambda_2(t)$ is a linear function of t, it has at most one zero-crossing point in the time interval $[\tau_0,\tau_f]$. Therefore, the optimal control $u^*(t)$ is piecewise-constant, with *at most one switching time*, and is given by $u^*(t) = -a_{min}$ for $t \in [\tau_0,\tau_s)$ and $u^*(t) = -a_{max}$ for $t \in [\tau_s,\tau_f]$, where τ_s is the unknown switching time. Using the expression for $u = u^*(t)$ and direct integration of (13) under the given initial conditions, $x_1(t)$ and $x_2(t)$ can be expressed as a function of τ_s. The solution for the end time τ_f that satisfies $x_1(\tau_f) = 0$ is then given by $\tau_f = \tau_s + [\nu - a_{min}\tau_s + \sqrt{(\nu - a_{min}\tau_s)^2 + 2a_{max}(\nu\tau_s - a_{min}\tau_s^2/2)}]/a_{max}$. Finally, the terminal velocity $x_{2f} = x_2(\tau_f)$ is obtained as $x_{2f} = -\sqrt{(\nu - a_{min}\tau_s)^2 + 2a_{max}(\nu\tau_s - a_{min}\tau_s^2/2)}$. Expressing the cost function $P(x_f,\tau_f)$ in terms of τ_s and using elementary calculus, it can then be shown that the maximal cost is attained for the critical switching time $\tau_s^* = \kappa\frac{\nu}{a_{min}}$, where κ is given in (10). Substitution of $\tau_s = \tau_s^*$ into the expression for P then gives the maximal value of $x_4^\infty - x_{40}$ in (10), which completes the proof. □

It is interesting to examine the expression for x_{4max} in (10) for two special cases. First, in case where $\gamma = 0$, i.e. x_4 is not changed by impacts, the optimal solution is the one that maximizes the Zeno time (Theorem 2), as x_4 is changing only during the continuous-time dynamics. The second special case is $b = 0$, i.e. the changes in x_4 are only due to impacts and are proportional to the impact velocities v_j. In this case, the optimal solution is the one that maximizes the sequence v_j (as in Theorem 1).

3.2 Bound on the Zeno value of x_3 in the **SVBB2** *system*

We now find an upper bound on the zeno value of x_3, denoted x_3^∞. Since the evolution of x_3 in (6) is only additive, one only needs to maximize the difference $x_3^\infty - x_{30}$. Unfortunately, the optimal solution of (6) for maximizing $x_3^\infty - x_{30}$ does not satisfy a scaling property as in (11). The reason is that $x_3^\infty - x_{30}$ depends on both ν and x_{40}, but the **SVBB2** system is homogeneous only with respect to multiplying x_2 and x_4 by the *same* scalar c, but not by unequal scalars. Thus, the exact tight bound cannot be found by solving a single optimal control problem on the first time interval, as done above for x_4^∞. Therefore, we only present here a *conservative* non-tight upper bound on x_3^∞, which is summarized in the following lemma.

Lemma 1. *Consider the* **SVBB2** *system in (6) under initial condition $x_0 = (0,\nu,x_{30},x_{40})^T$, where $\nu > 0$ and $x_{40} \geq 0$. Assuming that condition (7) is satisfied, all solutions are Zeno, and an upper bound on the value of x_3^∞ over all possible solutions is given by*

$$x_3^\infty - x_{30} < \frac{2\nu}{a_{min}}\left(\frac{x_{40}}{1-e'} + \frac{\beta(1+e') + e\alpha\gamma}{(1+e')(1-e')^2}\nu\right), \tag{14}$$

where $e' = e\sqrt{\alpha}$, $\alpha = \dfrac{a_{max}}{a_{min}}$, and $\beta = \dfrac{b}{a_{min}}$.

Proof: Consider a solution $\phi(t,j)$ of (6), and let $x_i(t,j)$ denote its x_i-component. Let us denote $v_j = x_2(t_j,j)$, $z_j = x_3(t_j,j)$ and $\omega_j = x_4(t_j,j)$. Additionally, denote $\delta_j = t_{j+1} - t_j$, which is the time difference between consecutive impacts. Using this notation, direct integration of (6) gives

$$\omega_k = x_{40} + \sum_{j=0}^{k-1} b\delta_j + \gamma\frac{v_{j+1}}{e} \tag{15}$$

$$z_k = x_{30} + \sum_{j=0}^{k-1} \omega_j\delta_j + \frac{1}{2}b\delta_j^2 \tag{16}$$

Using the relation (8), one obtains an upper bound on the sequence v_j as $v_j \leq \nu(e\sqrt{\alpha})^j$. On the other hand, it is straightforward to show that the maximal time difference δ_j is attained by selecting the constant acceleration $a = a_{min}$, hence one obtains $\delta_j \leq 2v_j/a_{min}$. Substituting the bounds on v_j and δ_j into (15) gives $\omega_j \leq z_{40} + (2\beta + \gamma\sqrt{\alpha})\frac{1-(e')^j}{1-e'}\nu$, where β and e' are defined in (14). Finally, substituting all the bounds into (16) and taking the limit as $k \to \infty$ gives the upper bound on the Zeno value of x_3 in (14). Note that the source of conservatism in this bound is that a solution with $a = a_{min}$ is selected for maximizing δ_j, whereas a different solution with a switch in a is selected for maximizing v_j. □

4. APPLICATION TO THE **BBSS** SYSTEM

In this section, we demonstrate the applicability of the results to Lagrangian hybrid systems by proving Zenoness of solutions of the **BBSS** system and deriving bounds on their Zeno limit point. Using the coordinates $q = (h, z)$, the state vector is given by $x = (h, \dot{h}, z, \dot{z})$, and the continuous- and discrete-time dynamics are given in (4) and (5). In the following, we focus on solutions in the neighborhood of an arbitrary point of Zeno equilibrium $x^* = (0, 0, z^*, \dot{z}^*)$. A necessary condition for x^* to attract Zeno solutions is that $\ddot{h}(z^*, \dot{z}^*) < 0$ (Or and Ames [2010], Lamperski and Ames [2008]). For convenience, we further assume that $\cos(z^*) < 0$ and $\dot{z}^* > 0$, so that both z and $\dot{z}$ are monotonically increasing along solutions in the vicinity of x^*. Next, we define a neighborhood Ω of x^* as

$$\Omega = \{(h, \dot{h}, z, \dot{z}) : \quad h \leq \epsilon_1, \ |\dot{h}| \leq \epsilon_2, \\ |z - z^*| \leq \epsilon_3, \text{ and } |\dot{z} - \dot{z}^*| \leq \epsilon_4\}. \tag{17}$$

Given this neighborhood, we define the following scalars

$$\begin{aligned} a_{min} &= \min_{(q,\dot{q})\in\Omega} -\ddot{h}(q, \dot{q}) \\ a_{max} &= \max_{(q,\dot{q})\in\Omega} -\ddot{h}(q, \dot{q}) \\ \gamma &= \max_{|z-z^*|<\epsilon_1} -(1+e)\frac{\cos(z)}{1+\cos(z)^2} \end{aligned} \tag{18}$$

A key observation is that when x lies within Ω, its hybrid dynamics is bounded that of the **SVBB2** system in (6). The conditions under which the solution is guaranteed to be Zeno and stays within Ω are summarized in the following Corollary.

Corollary 1. *Consider a hybrid solution $\phi(t, j)$ of the* **BBSS** *system under initial condition $x_0 = (0, \nu, z^*, \dot{z}^*)$ where $\nu, \dot{z}^* > 0$. Given $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 > 0$, define the neighborhood Ω as in (17). Assuming that $e^2\alpha < 1$, where $\alpha = \frac{a_{max}}{a_{min}}$ and a_{min}, a_{max} are defined in (18), if ν satisfies:*

$$\begin{aligned} &\nu < \min\{\mu_1, \ \mu_2, \ \mu_3, \ \mu_4, \ \mu_t\} \text{ , where} \\ &\mu_1 = \sqrt{2a_{min}\epsilon_1} \text{ , } \mu_2 = \epsilon_2/\sqrt{\alpha} \\ &\mu_3 = \frac{1-e^2\alpha}{2e\alpha\gamma}\left(-\dot{z}^* + \sqrt{(\dot{z}^*)^2 + 2z^* a_{min}(1-e\sqrt{\alpha})\epsilon_3}\right) \\ &\mu_4 = \frac{1-e\sqrt{\alpha}}{\gamma\sqrt{\alpha}}\epsilon_4 \text{ , and } \mu_t = \frac{1-e^2\alpha}{2(1+e)}a_{min}\epsilon_t \end{aligned} \tag{19}$$

Then $\phi(t, j)$ must be a Zeno solution such that its Zeno limit point lies in Ω and its Zeno time satisfies $T(\phi) < \epsilon_t$.

Proof: First, note that the initial condition x_0 lies within the neighborhood Ω. Assume that the solution $\phi(t, j)$ eventually leaves Ω. That is, there exist t', j' such that $\phi(t', j') \notin \Omega$, and that $\phi(t, j) \in \Omega$ for all $(t, j) \in \text{dom } \phi$ such that $j \leq j'$ and $t \leq t'$. We now show that condition (19) implies that $\phi(t', j') \in \Omega$, in contradiction with this assumption. The key observation is that since the solution $\phi(t, j)$ of the **BBSS** system lies within Ω for $t < t'$ and $j \leq j'$, it is also a solution of the **SVBB2** system (6) with the parameters a_{min}, a_{max} and γ defined as in (18) and $b = 0$. First, since $e\sqrt{\alpha} < 1$, inequality (8) implies that the sequence of $v_j = x_2(t_j, j)$ is decaying for $j \leq j'$, so that $v_{j'} < \nu$. Moreover, it can be shown that $x_1(t, j) \leq v_j^2/2a_{min}$, for all $(t, j) \in \text{dom } \phi$. Therefore, the bound $\nu < \mu_1$ in (19) implies that $x_1(t', j') < \epsilon_1$. Second, using the proof of Theorem 1, it can be shown that $|x_2(t, j)| < \sqrt{\alpha}v_j$ for all $(t, j) \in \text{dom } \phi$. Since $v_{j'} \leq \nu$, the bound $\nu < \mu_2$ in (19) implies that $x_2(t', j') < \epsilon_2$. Third, since x_3 is monotonically increasing in t, one concludes that $x_3(t', j')$ is less than the bound on x_3^∞ in (14). Substituting $\beta = 0$, $x_{30} = z^*$ and $x_{40} = \dot{z}^*$, the bound $\nu < \mu_3$ in (19) implies that $x_3(t', j') - z^* < \epsilon_3$. Fourth, since x_4 is non-decreasing in t, one concludes that $x_4(t', j')$ is less than the bound on x_4^∞ in (10). Substituting $\beta = 0$ and $x_{40} = \dot{z}^*$, the bound $\nu < \mu_4$ in (19) then implies that $0 < x_3(t', j') - z^* < \epsilon_4$. Using the definition of Ω in (17), we now conclude that $\phi(t', j') \in \Omega$, in contradiction to the original assumption. Finally, using Theorem 1, the condition $e^2\alpha < 1$ implies that $\phi(t, j)$ is Zeno, and Theorem 2 implies that its Zeno time $T(\phi)$ satisfies the bound in (9). Therefore, the bound $\nu < \mu_t$ in (19) implies that $T(\phi) < \epsilon_t$. □

Simulation Example: We now show a simulation example for the **BBSS** system. The system parameters are chosen as $m=1$, $g=1$ and $e=0.5$. We simulate the system under initial condition $x_0 = (1 \ 0 \ 2 \ 0)^T$. At each impact, we define the neighborhood Ω about the post-impact point $\phi(t_j, j)$ according to (17), with $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_t = 0.001$, and compute a_{min}, a_{max} and γ according to (18). The simulation is stopped when the post-impact velocity $\dot{h}$ satisfied the condition (19), which guarantees Zenoness of the solution and boundedness of the Zeno point within Ω. For the given initial condition, simulation was stopped after 13 impacts. Fig. 2(a)-(c) shows time plots of h,z and $\dot{z}$, where the thick dots denote discrete values at the impact times. Fig. 2(d) plots the trajectory of the ball in (z, y)-plane, overlayed with the constraint surface $y = \sin(z)$.

Remarks: In this section we focused on a system with two DOF, and assumed that $\dot{z}$, $\ddot{z}$ and γ are all positive. The extension to more general Lagrangian hybrid systems is outlined as follows. First, cases where $\dot{z}$, $\ddot{z}$ and γ are not sign-definite within Ω can be treated as well, by placing bounds on the absolute values $|\dot{z}|$, $|\ddot{z}|$ and $|\gamma|$ and using the triangle inequality. Nevertheless, this will make the obtained bounds much more conservative in case of sign reversal along solutions. Second, the results can be applied also to systems with multiple DOF, by deriving a separate bound on the dynamics of each unconstrained coordinate z_i. Alternatively, one could find bounds on the evolution of the magnitude $||z - z^*||$ and $||\dot{z} - \dot{z}^*||$ by deriving norm bounds on $\ddot{z}$ and γ and using the triangle inequality as done in (Or and Ames [2010]).

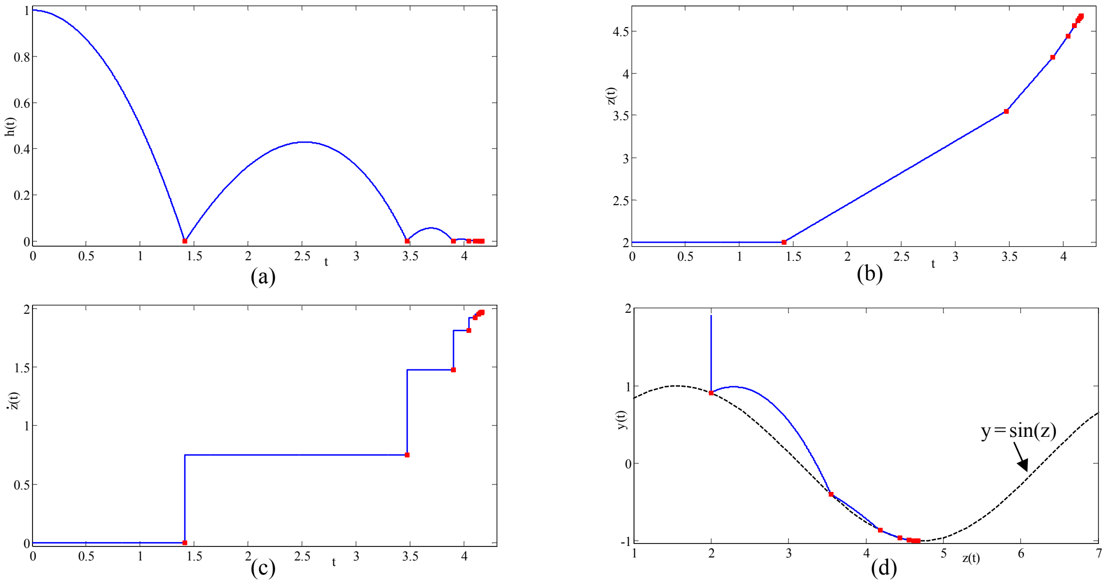

Fig. 2. Simulation results of the **BBSS** system, the thick dots mark values at impact times. (a) h vs. time t. (b) z vs. time t. (c) $\dot{z}$ vs. time t. (d) $y(t)$ vs. $z(t)$ overlayed with the constraint surface $y = \sin(z)$.

5. CONCLUSION

We studied the hybrid dynamics of the set-valued bouncing ball with an additional unconstrained coordinate. We derived bounds on the drift in the unconstrained coordinate along Zeno solutions. The results can be applied to obtaining conditions under which the solution of a 2-DOF Lagrangian hybrid system is Zeno and derivation of explicit bounds on the unknown Zeno point and Zeno time, as demonstrated on the **BBSS** system. Two future directions for extension of the results are as follows. First, consideration of set-valued dynamics that are more complicated than the linear dynamics of the **SVBB** may enable derivation of tighter bounds on Zeno solutions of Lagrangian hybrid systems. Second, generalization of the results for proving stability of hybrid periodic orbits with Zeno behavior (e.g. Bourgeot and Brogliato [2005], Or and Ames [2009]) is a challenging open problem.

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