Supplementary document to the Letter: "Asymmetry and stability of shape kinematics in microswimmers motion"

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This supplementary document begins with systematic formulation of the equations of motion for a general articulated swimmer in viscous fluid. Then explicit formulation for the particular model of Purcell's three-link swimmer is given. Finally, detailed proofs of Results 1 to 4 from the Letter are given.

I. FORCE- AND SHAPE- CONTROLLED ROBOTIC MICROSWIMMER

Consider an articulated robotic swimmer in a viscous fluid. It is assumed that the swimmer is very small, or slow, or the fluid is highly viscous, so that the Reynolds number is nearly zero. This implies that the fluid is governed by *Stokes equations* and inertial effects can be neglected so that the motion is quasistatic [1]. The robot consists of n rigid links which are connected by m joints. The joints are actuated by prescribing either their motion or their internal force or torque. The robot may consist of several open (i.e. serial) kinematic chains, but no closed chains. Let $\mathbf{r}_b$ denote the origin of a body-fixed reference frame which is rigidly attached to the swimmer. Let us attach a moving frame to the ith link, and denote the position of its origin by $\mathbf{r}_i$. The joints, which are enumerated by $\mathcal{J} = \{1 \dots m\}$ are divided into a set of linear joints $\mathcal{J}_{lin} \subseteq \mathcal{J}$ and rotary joints $\mathcal{J}_{rot} \subseteq \mathcal{J}$. A linear joint $j \in \mathcal{J}_{lin}$ imposes linear relative motion between links along the direction of the unit vector $\hat{\mathbf{l}}_j$, with the linear velocity denoted by u_j. A rotary joint $j \in \mathcal{J}_{rot}$ imposes relative rotation between links about an axis whose direction is given by the unit vector $\hat{\mathbf{l}}_j$, where the angular velocity is denoted by u_j and $\mathbf{b}_j$ denotes a point on the axis. The topological structure of the kinematic chains comprising the swimmer is encoded by the indicator matrix I_{ij}, such that $I_{ij} = 1$ if the location of the ith link is affected by the jth joint, and $I_{ij} = 0$ otherwise. Let $\mathbf{v}_i$ and $\boldsymbol{\omega}_i$ denote the linear angular and velocity of the ith link, respectively, and define $\mathbf{V}_i = \begin{pmatrix} \mathbf{v}_i \\ \boldsymbol{\omega}_i \end{pmatrix}$. Similarly, let $\mathbf{V}_b = \begin{pmatrix} \mathbf{v}_b \\ \boldsymbol{\omega}_b \end{pmatrix}$ denote the linear and angular velocity of the body-fixed reference frame. The kinematic relation between the body velocity, the joints velocities and the links velocities is given by

$$\begin{aligned} \mathbf{v}_i &= \mathbf{v}_b + \boldsymbol{\omega}_b \times (\mathbf{r}_i - \mathbf{r}_b) + \sum_{j\in\mathcal{J}_{lin}} I_{ij} u_j \hat{\mathbf{l}}_j + \sum_{j\in\mathcal{J}_{rot}} I_{ij} u_j \hat{\mathbf{l}}_j \times (\mathbf{r}_i - \mathbf{b}_j) \\ \boldsymbol{\omega}_i &= \boldsymbol{\omega}_b + \sum_{j\in\mathcal{J}_{rot}} I_{ij} u_j \hat{\mathbf{l}}_j. \end{aligned} \tag{1}$$

The kinematic relations (1) can be written in matrix form as

$$\mathbf{V} = \mathbf{T}\mathbf{V}_b + \mathbf{E}\mathbf{u}, \tag{2}$$

where

$$\begin{aligned} &\mathbf{V} = \begin{pmatrix} \mathbf{V}_1 \\ \vdots \\ \mathbf{V}_n \end{pmatrix}, \ \mathbf{T} = \begin{pmatrix} \mathbf{T}_1 \\ \vdots \\ \mathbf{T}_n \end{pmatrix}_{6n\times 6}, \ \mathbf{E} = \begin{pmatrix} \mathbf{E}_{11} & \dots & \mathbf{E}_{1m} \\ \vdots & \vdots & \vdots \\ \mathbf{E}_{n1} & \dots & \mathbf{E}_{nm} \end{pmatrix}_{6n\times m}, \ \mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \\ &\mathbf{T}_i = \begin{pmatrix} \mathbf{I}_{3\times 3} & -[(\mathbf{r}_i - \mathbf{r}_b)\times] \\ \mathbf{O}_{3\times 3} & \mathbf{I}_{3\times 3} \end{pmatrix}, \ \mathbf{E}_{ij} = \begin{cases} I_{ij} \begin{pmatrix} \hat{\mathbf{l}}_j \\ \vec{0} \end{pmatrix} & j \in \mathcal{J}_{lin} \\ I_{ij} \begin{pmatrix} \hat{\mathbf{l}}_j \times (\mathbf{r}_i - \mathbf{b}_j) \\ \hat{\mathbf{l}}_j \end{pmatrix} & j \in \mathcal{J}_{rot} \end{cases} \end{aligned} \tag{3}$$

where $\mathbf{I}$ is the identity matrix and $[\mathbf{a}\times]$ is the cross-product matrix which satisfies $[\mathbf{a}\times]\mathbf{b} = \mathbf{b} \times \mathbf{b}$ for all $\mathbf{a}, \mathbf{b} \in I\!R^3$.

Next, we consider the forces and moments acting on the swimmer's links. Let $\mathbf{f}_i$ and $\mathbf{m}_i$ denote the force and moment exerted by the fluid on the ith link, respectively. Let us also denote $\mathbf{F}_i = \begin{pmatrix} \mathbf{f}_i \\ \mathbf{m}_i \end{pmatrix}$ and $\mathbf{F} = \begin{pmatrix} \mathbf{F}_1 \\ \vdots \\ \mathbf{F}_n \end{pmatrix}$. A fundamental property of viscous flow, which stems from the linearity of Stokes equation, is the existence of a linear relationship between hydrodynamic forces and velocities, which is given by

$$\mathbf{F} = \boldsymbol{\mathcal{R}}\mathbf{U}. \tag{4}$$

The matrix $\boldsymbol{\mathcal{R}}$ is called the *resistance tensor*, and depends on the positions of all the links. The net force and moment acting on the body-fixed frame, denoted by $\mathbf{f}_b$ and $\mathbf{m}_b$ respectively, are given by

$$\mathbf{f}_b = \sum_{i=1}^{n} \mathbf{f}_i \ , \qquad \mathbf{m}_b = \sum_{i=1}^{n} \mathbf{m}_i + (\mathbf{r}_i - \mathbf{r}_b) \times \mathbf{f}_i. \tag{5}$$

It can be verified that the relation (5) can be written in matrix form as

$$\mathbf{F}_b = \begin{pmatrix} \mathbf{f}_b \\ \mathbf{m}_b \end{pmatrix} = \mathbf{T}^T \mathbf{F}, \tag{6}$$

where the matrix $\mathbf{T}$ defined in (3) is precisely the same as the one in (2). Since the motion in Stokes flow is quasi-static, the forces and torques on the body must vanish $\mathbf{F}_b = 0$. Substituting (4) and (2) into (6) yields

$$\mathbf{F}_b = \mathbf{T}^T \boldsymbol{\mathcal{R}} (\mathbf{T} \mathbf{V}_b + \mathbf{E} \mathbf{u}) = 0. \tag{7}$$

Inverting the relation (7) then gives a linear relation between the rigid-body velocity and the joints velocities of the swimmer, as

$$\mathbf{V}_b = -(\mathbf{T}^T \boldsymbol{\mathcal{R}} \mathbf{T})^{-1} \mathbf{T}^T \boldsymbol{\mathcal{R}} \mathbf{E} \mathbf{u}. \tag{8}$$

Assuming that the joints velocities $\mathbf{u}$ are directly controlled by the robot, Equation (8) gives the resulting swimmer's body velocity $\mathbf{V}_b$ for any prescribed value of $\mathbf{u}$. Note that equation (7) can also be written as

$$\mathbf{F}_b = \boldsymbol{\mathcal{R}}_{bb} \mathbf{V}_b + \boldsymbol{\mathcal{R}}_{bu} \mathbf{u} = 0, \text{ where } \boldsymbol{\mathcal{R}}_{bb} = \mathbf{T}^T \boldsymbol{\mathcal{R}} \mathbf{T} \text{ and } \boldsymbol{\mathcal{R}}_{bu} = \mathbf{T}^T \boldsymbol{\mathcal{R}} \mathbf{E}. \tag{9}$$

The interpretation of (9) is as follows. The matrix $\boldsymbol{\mathcal{R}}_{bb}$ gives the relation between force and moment acting on the body and its rigid-body velocity, assuming that all joints are locked (i.e. "dragging resistance" [2]). The matrix $\boldsymbol{\mathcal{R}}_{bu}$ gives the relation between force and moment acting on the body and the joints velocities, assuming that the body frame is held fixed (i.e. "pumping resistance" [2]).

Force-controlled swimmer: Another important possible scenario is when the robot does not directly control its joints velocities $\mathbf{u}$, but instead it prescribes the *internal forces and moments* supplied at the actuated joints. In order to study this case, we need to consider additional force-and-moment balance on a section of the swimmer, which is cut at the jth joint, as follows. The partial kinematic chain composed of the links with indices $\mathcal{I}_j = \{i : I_{ij} = 1\}$ is subjected to force and moment exerted by the rest of the robot through the

jth joint, which are denoted by $\tilde{\mathbf{f}}^j$ and $\tilde{\mathbf{m}}^j$, respectively. Due to static equilibrium of the partial kinematic chain, these force and moment are balanced by the hydrodynamic forces and moments $\mathbf{f}_i$ and $\mathbf{m}_i$ for $i \in \mathcal{I}_j$, giving rise to the following relations:

$$\tilde{\mathbf{f}}^j = -\sum_{i \in \mathcal{I}_j} \mathbf{f}_i \quad , \quad \tilde{\mathbf{m}}^j = -\sum_{i \in \mathcal{I}_j} (\mathbf{m}_i + (\mathbf{r}_i - \mathbf{b}_j) \times \mathbf{f}_i) \ \ , j = 1 \ldots m. \tag{10}$$

The controlled force or moment at each joint, denoted by τ_j, is directed along the joint axis, and is given by

$$\tau_j = \begin{cases} \hat{\mathbf{l}}_j \cdot \tilde{\mathbf{f}}^j & j \in \mathcal{J}_{lin} \\ \hat{\mathbf{l}}_j \cdot \tilde{\mathbf{m}}^j & j \in \mathcal{J}_{rot} \end{cases} . \tag{11}$$

Denoting $\boldsymbol{\tau} = (\tau_1 \ldots \tau_m)^T$ as the vector of controlled forces or moments at the actuated joints, it can be verified that the projection of the balance equation (10) along the relevant direction for the jth joint according to (11) gives rise to the matrix-form equation

$$\boldsymbol{\tau} = -\mathbf{E}^T \mathbf{F}, \tag{12}$$

where the matrix $\mathbf{E}$ defined in (3) is precisely the same matrix used in the kinematic relation (2). The fact that the matrices $\mathbf{T}^T$ and $\mathbf{E}^T$ in the force-and-moment balances (6) and (12) are transpose of the matrices in the kinematic relation (2) is a standard result in formulation of kinematics and statics of serial robots which is a consequence of the principle of virtual work, cf. [3, 4], where $\mathbf{T}$ and $\mathbf{E}$ can be interpreted as the robot's Jacobian matrices. Next, substituting (2), (4) and (8) into (12) gives

$$\boldsymbol{\tau} = -\mathbf{E}^T \boldsymbol{\mathcal{R}} \mathbf{V} = -\mathbf{E}^T \boldsymbol{\mathcal{R}} (\mathbf{T}\mathbf{V}_b + \mathbf{E}\mathbf{u}) = \left(\mathbf{E}^T \boldsymbol{\mathcal{R}} \mathbf{T} (\mathbf{T}^T \boldsymbol{\mathcal{R}} \mathbf{T})^{-1} \mathbf{T}^T \boldsymbol{\mathcal{R}} \mathbf{E} - \mathbf{E}^T \boldsymbol{\mathcal{R}} \mathbf{E} \right) \mathbf{u}. \tag{13}$$

This equation can then be inverted in order to obtain $\mathbf{u}$ as a function of $\boldsymbol{\tau}$. Finally, substituting the expression for $\mathbf{u}$ from (13) into (8) gives

$$\mathbf{V}_b = -(\mathbf{T}^T \boldsymbol{\mathcal{R}} \mathbf{T})^{-1} \mathbf{T}^T \boldsymbol{\mathcal{R}} \mathbf{E} \left(\mathbf{E}^T \boldsymbol{\mathcal{R}} \mathbf{T} (\mathbf{T}^T \boldsymbol{\mathcal{R}} \mathbf{T})^{-1} \mathbf{T}^T \boldsymbol{\mathcal{R}} \mathbf{E} - \mathbf{E}^T \boldsymbol{\mathcal{R}} \mathbf{E} \right)^{-1} \boldsymbol{\tau}. \tag{14}$$

This equation gives the relation between the controlled forces/moments at the joints to the body velocity of the swimmer. A more concise formulation of equation (14) is given by

$$\mathbf{V}_b = -\boldsymbol{\mathcal{R}}_{bb}^{-1} \boldsymbol{\mathcal{R}}_{bu} \left(\boldsymbol{\mathcal{R}}_{bu}^T \boldsymbol{\mathcal{R}}_{bb}^{-1} \boldsymbol{\mathcal{R}}_{bu} + \boldsymbol{\mathcal{R}}_{uu} \right)^{-1} \boldsymbol{\tau} \qquad \text{where } \boldsymbol{\mathcal{R}}_{uu} = -\mathbf{E}^T \boldsymbol{\mathcal{R}} \mathbf{E} \tag{15}$$

and $\boldsymbol{\mathcal{R}}_{bb}, \boldsymbol{\mathcal{R}}_{bu}$ are defined in (9). $\boldsymbol{\mathcal{R}}_{uu}$ can be interpreted as the forces/moments exerted on the joints axes by the fluid (i.e. $-\boldsymbol{\tau}$) under given joints velocities $\mathbf{u}$, assuming that the body frame is held fixed.

Possible reductions of the general formulation above can be obtained in some simplified cases, as follows. First, consider the case of a planar swimmer which translates in xy plane while all rotations are along the z-axis. In this case, one can reduce the linear velocities $\mathbf{v}_i, \mathbf{v}_b$ and linear forces $\mathbf{f}_i, \mathbf{f}_b$ to their (x, y)-components and the angular velocities $\boldsymbol{\omega}_i, \boldsymbol{\omega}_b$ and moments $\mathbf{m}_i, \mathbf{m}_b$ to their z-component, leading to twofold reduction in the dimension of the resistance matrix $\boldsymbol{\mathcal{R}}$ to $3n \times 3n$. The matrices $\mathbf{T}$ and $\mathbf{E}$ in (3) reduce to dimensions $3n \times 3$ and $3n \times m$, respectively. The matrix $[(\mathbf{r}_i - \mathbf{r}_b)\times]$ in $\mathbf{T}$ is replaced with the row vector $(\bar{\mathbf{r}}_i - \bar{\mathbf{r}}_b)^T \mathbf{J}^T$ and the column vector $\hat{\mathbf{l}}_j \times (\mathbf{r}_i - \mathbf{b}_j) = \hat{\mathbf{z}} \times (\mathbf{r}_i - \mathbf{b}_j)$ is replaced with $\mathbf{J}(\bar{\mathbf{r}}_i - \bar{\mathbf{b}}_j)$, where $\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\bar{\mathbf{r}}_i, \bar{\mathbf{r}}_b, \bar{\mathbf{b}}_j$ are the (x, y)-components of $\mathbf{r}_i, \mathbf{r}_b, \mathbf{b}_j$, respectively.

Second, consider the case where the hydrodynamic interaction between the rigid links is assumed negligible, such as, for example, when the swimmer consists of slender links [5]. In this case, the resistance tensor becomes block-diagonal $\boldsymbol{\mathcal{R}} = \text{diag}(\boldsymbol{\mathcal{R}}_1, \ldots, \boldsymbol{\mathcal{R}}_n)$, so that the force-velocity relation for the ith link is $\mathbf{F}_i = \boldsymbol{\mathcal{R}}_i \mathbf{V}_i$. Thus, the resistance matrices $\boldsymbol{\mathcal{R}}_{bb}, \boldsymbol{\mathcal{R}}_{bu}$ and $\boldsymbol{\mathcal{R}}_{uu}$ defined in (9) and (15) reduce to

$$\boldsymbol{\mathcal{R}}_{bb} = \sum_{i=1}^{n} \mathbf{T}_i^T \boldsymbol{\mathcal{R}}_i \mathbf{T}_i \ , \ \boldsymbol{\mathcal{R}}_{bu} = \sum_{i=1}^{n} \mathbf{T}_i^T \boldsymbol{\mathcal{R}}_i \mathbf{E}_i \text{ and } \boldsymbol{\mathcal{R}}_{uu} = -\sum_{i=1}^{n} \mathbf{E}_i^T \boldsymbol{\mathcal{R}}_i \mathbf{E}_i \ , \text{ where } \mathbf{E}_i = (\mathbf{E}_{i1} \ldots \mathbf{E}_{im}). \tag{16}$$

II. EQUATIONS OF MOTION OF PURCELL'S THREE-LINK SWIMMER

A sketch of Purcell's three swimmer model appears in Figure 1(a). It consists of three rigid links of lengths l_0, l_1, l_2 and two revolute joints actuated by internal torques. The shape of the swimmer is described by the two relative angles between the links, denoted by $\mathbf{s} = (\phi_1, \phi_2)^T$. The state of the swimmer is described by the planar position and orientation of a reference frame which is attached to the central link, and is denoted by $\mathbf{q} = (x, y, \theta)^T$. It is assumed that the swimmer is submerged in an unbounded fluid domain of viscosity μ, and that is motion is confined to the xy plane.

In order to obtain an explicit formulation of the swimmer's dynamic equation of motions, the simplest available model of *resistive force theory* for slender bodies [5, 6] will be used.

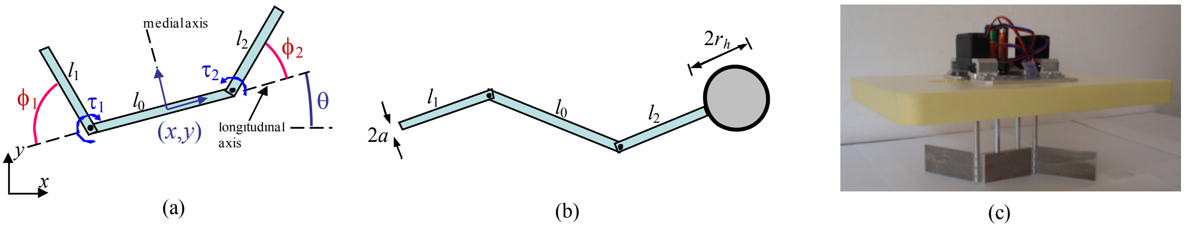

FIG. 1: (a) Purcell's three-link swimmer model. (b) The three link swimmer with a spherical head. (c) Robotic three-link swimmer built at the author's laboratory.

This theory approximates the hydrodynamic forces and torques exerted by a viscous fluid on a moving slender body, as follows. Consider a slender filament of length l and circular cross-section of radius a. Let $\mathbf{v}(s)$ denote the instantaneous local velocity of the filament at a location s along its axis. This velocity can be decomposed into its axial and normal components as $\mathbf{v}(s) = \mathbf{v}_t(s) + \mathbf{v}_n(s)$, where $\mathbf{v}_t(s) = (\mathbf{e}_t(s) \cdot \mathbf{v}(s))\mathbf{e}_t(s)$ and $\mathbf{e}_t(s)$ is a unit vector tangent to the axis of the filament at location s. The basic result of slender body theory states that to leading order in $\ln(l/a)$, the local force acting at location s is given by $\mathbf{f}(s) = -c_n\mathbf{v}_n(s) - c_t\mathbf{v}_t(s)$ where $c_t = 2\pi\mu/\ln(l/a)$ and $c_n = 2c_t$. Next, it is assumed that the filament is a straight rigid rod whose motion is confined to a plane. Let $\mathbf{v}$ and ω denote the linear and angular velocities of a reference frame located at the center of the rod, and let $\mathbf{e}_t$ denote be a unit vector along the rod's axis. It is then straightforward to show (cf. [7]) that the net force and torque acting on the rod is $\mathbf{f} = -c_t l\mathbf{v}_t - c_n l\mathbf{v}_n$ and $\tau = -c_n l^3/12\omega$, where $\mathbf{v}_t = (\mathbf{v} \cdot \mathbf{e}_t)\mathbf{e}_t$ and $\mathbf{v}_n = \mathbf{v} - \mathbf{v}_t$, and τ is taken with respect to the center of the rod. The swimmer consists of three links, labeled $0, 1$ and 2, and it is assumed that the three links do not interact hydrodynamically, which is correct to leading order for slender bodies. Using the notation defined in the previous section, the 3×3 resistance matrices $\boldsymbol{\mathcal{R}}_i$ for planar motion of the three links are given by

$$\boldsymbol{\mathcal{R}}_i = -\tfrac{2\pi\mu}{\ln(l_i/a)} l_i \begin{pmatrix} 1+\sin^2(\alpha_i) & -\cos(\alpha_i)\sin(\alpha_i) & 0 \\ -\cos(\alpha_i)\sin(\alpha_i) & 1+\cos^2(\alpha_i) & 0 \\ 0 & 0 & l_i^2/6 \end{pmatrix} \tag{17}$$

where $\alpha_0 = \theta$, $\alpha_1 = \theta - \phi_1$, and $\alpha_2 = \theta + \phi_2$.

The matrices $\mathbf{T}_i$ and $\mathbf{E}_i$, which encode the geometric structure of the swimmer and its actuated joints, are given by

$$\mathbf{T}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{T}_1 = \begin{pmatrix} 1 & 0 & \frac{1}{2}(l_0 \sin\theta + l_1 \sin\alpha_1) \\ 0 & 1 & -\frac{1}{2}(l_0 \cos\theta + l_1 \cos\alpha_1) \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{T}_2 = \begin{pmatrix} 1 & 0 & -\frac{1}{2}(l_0 \sin\theta + l_2 \sin\alpha_2) \\ 0 & 1 & \frac{1}{2}(l_0 \cos\theta + l_2 \cos\alpha_2) \\ 0 & 0 & 1 \end{pmatrix}$$
$$\mathbf{E}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{E}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \end{pmatrix}, \mathbf{E}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{18}$$

where α_1 and α_2 are defined in (17). Using (17) and (18), the expressions for $\boldsymbol{\mathcal{R}}_{bb}$, $\boldsymbol{\mathcal{R}}_{bu}$ and $\boldsymbol{\mathcal{R}}_{uu}$ can be formulated according to (16), and equations (13) and (15) which govern the swimmer's dynamics can be formulated. Note that $\mathbf{u}$ is precisely the vector of velocities of shape variables, $\mathbf{u} = \dot{\mathbf{s}}$. Thus, equations (13) and (15) can be written in the form

$$\dot{\mathbf{q}} = \mathbf{G}(\mathbf{q}, \mathbf{s})\dot{\mathbf{s}}. \tag{19}$$

$$\dot{\mathbf{s}} = \mathbf{H}(\mathbf{s})\boldsymbol{\tau}. \tag{20}$$

These are precisely equations (1) and (2) from the Letter.

Another case which is considered in the paper is where a spherical "head" with radius r_h is attached to the end of link 2, see Figure 1(b). For simplicity, the hydrodynamic interaction of the sphere and the slender links is neglected, so that the simple results of Stokes resistance of a single sphere [1] can be used. Therefore, the sphere can be incorporated into the swimmer's equations of motion by simply adding another rigid link with index $i = 3$, whose matrices are given by

$$\boldsymbol{\mathcal{R}}_3 = -\frac{1}{6\pi\mu r_h} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{3}{4r_h^2} \end{pmatrix}, \mathbf{T}_3 = \begin{pmatrix} 1 & 0 & -(\frac{1}{2}l_0 \sin\theta + (l_2 + r_h)\sin\alpha_2) \\ 0 & 1 & \frac{1}{2}l_0 \cos\theta + (l_2 + r_h)\cos\alpha_2 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{E}_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{21}$$

Next, simulation results that were omitted from the paper due to lack of space are presented. Consider the linear control input $\boldsymbol{\tau}(t)$ shown in Figure 2(a), which is reversible and axisymmetric. It results in a periodic shape kinematics shown in Figure 2(b) as a closed loop in (ϕ_1, ϕ_2)-plane. Figure 2(c) shows motion snapshots of the swimmer's state and shape during a complete period. It can be seen that the net motion of the swimmer

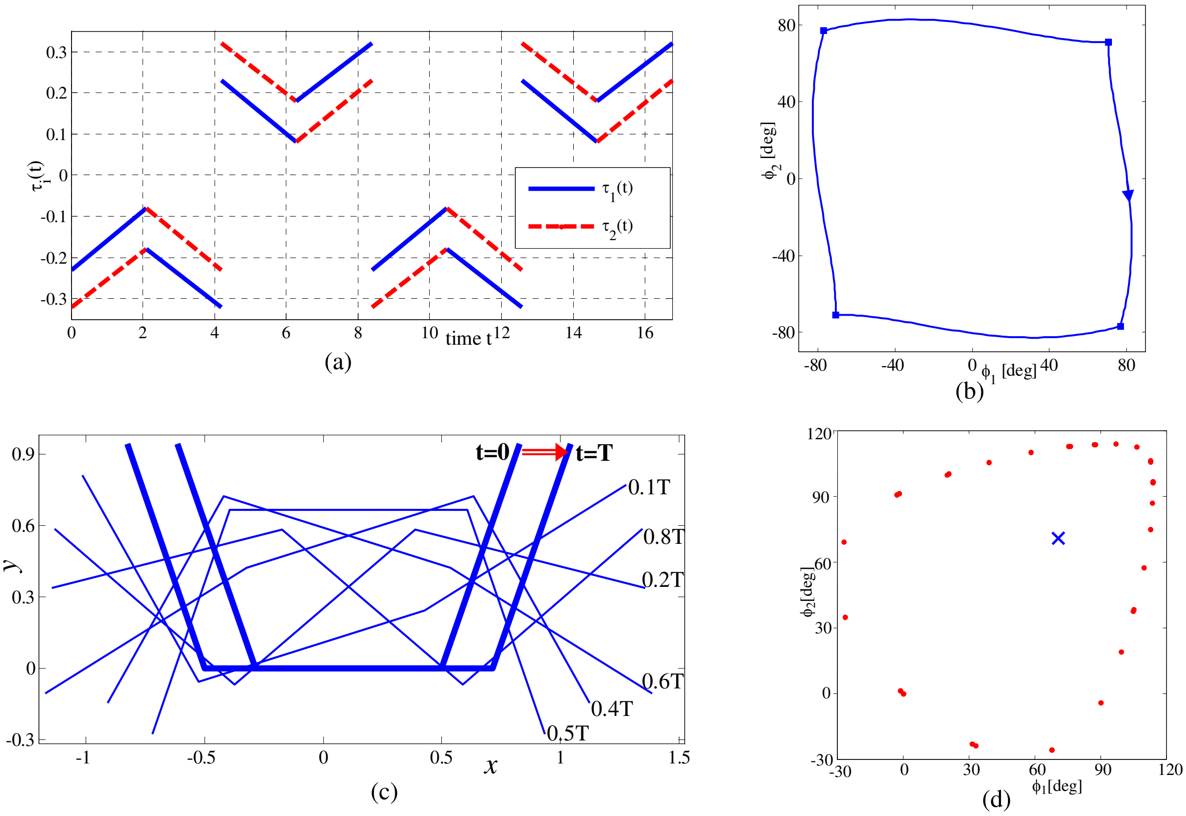

FIG. 2: (a) Linear time profile of joint torques $\tau_i(t)$. (b) The periodic orbit in (ϕ_1, ϕ_2)-plane under the linear time profile of $\boldsymbol{\tau}(t)$. (c) Snapshots of the swimmer under the linear time profile of $\boldsymbol{\tau}(t)$. (d) The invariant circle of $\mathbf{s}(kT)$ in (ϕ_1, ϕ_2)-plane for $\mathbf{s}(0) = (0, 0)^T$.

is indeed pure translation along the x direction. Next, simulation results under initial conditions $\mathbf{s}(0) = (0, 0)^T$ are shown in Figure 2(d). The points of $\mathbf{s}(kT)$ in (ϕ_1, ϕ_2)-plane for $k = 0, 1, \ldots, 30$ are plotted, which clearly form a closed curve encircling the fixed point $\mathbf{s}_e$ (marked by '$\times$'). This is precisely the invariant circle, which also respects the reversing symmetry, i.e. it is symmetric with respect to the line $\phi_1 = \phi_2$.

Finally, it is important to report that a robotic three-link swimmer prototype has been recently designed and constructed at the laboratory headed by the author, see Figure 1(c). The swimmer is placed in a highly viscous silicone fluid in order to guarantee very low Reynolds number hydrodynamics, and is mounted on a floatation foam cell in order to support the gravitational load and enforce motion in the horizontal plane. It is currently equipped with two servo motors which directly control the joint angles, and preliminary motion experiments are currently being conducted. In the future, it is planned to replace the servo motors with DC motors which can control the internal joint torques, in order to experimentally demonstrate the theoretical results presented in the Letter.

III. PROOFS OF THE MAIN RESULTS

This section gives the proofs of the main results 1 to 4 from the Letter. As a preliminary step, one needs to study the *symmetries* underlying the geometric structure of the swimmer's equations of motion. First, note that the swimmer possesses axisymmetry, i.e. reflection symmetry about the longitudinal axis of the central link, that is, (20) is invariant under reversing the sign of $\mathbf{s}$, $\dot{\mathbf{s}}$ and $\boldsymbol{\tau}$. This implies the relation

$$\mathbf{H}(-\mathbf{s}) = \mathbf{H}(\mathbf{s}). \tag{22}$$

Second, in case where $l_1 = l_2$, the swimmer also possesses front-back symmetry, which is a reflection symmetry about the bisecting line perpendicular to the central link — the medial axis (see Figure 1(a)). This implies that the equation (20) is invariant with respect to interchanging between the right and left links. This relation can be formulated as

$$\mathbf{H}(\mathbf{M}_s\mathbf{s}) = \mathbf{M}_s\mathbf{H}(\mathbf{s})\mathbf{M}_s, \text{ where } \mathbf{M}_s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{23}$$

Multiplication by $\mathbf{M}_s$ in (23) is simply interchanging between the left and right links' angles. The proof of (23) is based on the fact that if one defines $\mathbf{s}' = \mathbf{M}_s\mathbf{s}$ and $\boldsymbol{\tau}' = \mathbf{M}_s\boldsymbol{\tau}$, then the front-back symmetry implies that $\dot{\mathbf{s}}' = \mathbf{H}(\mathbf{s}')\boldsymbol{\tau}'$. Combining this with (20) then leads to (23).

Third, since the fluid domain is unbounded, equation (19) is invariant with respect to rigid-body motion, so that the swimmer's velocity $\dot{\mathbf{q}}$ expressed in body-fixed reference frame depends only on the shape $\mathbf{s}$. This relation, termed *gauge symmetry* (e.g. [8, 9]), implies that

$$\begin{aligned} &\mathbf{G}(\mathbf{q},\mathbf{s}) = \mathbf{D}(\theta)\mathbf{G}_0(\mathbf{s}), \text{ where} \\ &\mathbf{G}_0(\mathbf{s}) = \mathbf{G}(0,\mathbf{s}) \text{ and } \mathbf{D}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{24}$$

$\mathbf{G}_0$ is the matrix $\mathbf{G}$ evaluated at $\theta = 0$, i.e. when expressing $\dot{\mathbf{q}}$ in body-fixed reference frame. Multiplication by the matrix $\mathbf{D}(\theta)$ in (24) is simply a rotation by θ which transforms the body velocity vector $\dot{\mathbf{q}}$ from body-fixed reference frame to an inertial reference frame. Fourth, axisymmetry implies that

$$\mathbf{G}_0(-\mathbf{s}) = -\mathbf{M}_q\mathbf{G}_0(\mathbf{s}) \text{ , where } \mathbf{M}_q = \mathsf{diag}(1,-1,-1). \tag{25}$$

Multiplication by the matrix $\mathbf{M}_q$ in (25) is simply a reflection of the body velocity $\dot{\mathbf{q}}$ (expressed in body-fixed frame) about the longitudinal axis.

Finally, if $l_1 = l_2$, front-back symmetry implies that

$$\mathbf{G}_0(\mathbf{M}_s\mathbf{s}) = \mathbf{M}_l\mathbf{G}_0(\mathbf{s})\mathbf{M}_s \text{ , where } \mathbf{M}_l = \mathsf{diag}(-1,1,-1). \tag{26}$$

Multiplication by $\mathbf{M}_l$ in (26) is simply a reflection of the body velocity $\dot{\mathbf{q}}$ (expressed in body-fixed frame) about the medial axis.

It is important to note that the symmetries listed above are consequence of the geometric structure of the swimmer, and do not depend on the hydrodynamic model. That is, these relations also hold if a more accurate model of the hydrodynamic interactions is used instead of resistive force theory.

Next, a series of Lemmas are given, which will be useful for proving results 1 to 4.

Lemma S.1 (time invariance): If $\mathbf{s}^*(t)$ is a solution of (20) under initial condition $\mathbf{s}(0) = \mathbf{s}^*(0)$ and input $\boldsymbol{\tau}(t) = \boldsymbol{\tau}^*(t)$, then $\mathbf{s}^*(t+c)$ is a solution of (20) under initial condition $\mathbf{s}(0) = \mathbf{s}^*(c)$ and input $\boldsymbol{\tau}(t) = \boldsymbol{\tau}^*(t+c)$ for any $c \in I\!R$.

The proof of this lemma is straightforward, and it simply states that since (20) does not depend explicitly on time, the starting time can be shifted by any arbitrary constant.

Lemma S.2 (time reversal): If $\mathbf{s}^*(t)$ is a solution of (20) under initial condition $\mathbf{s}(0) = \mathbf{s}^*(0)$ and input $\boldsymbol{\tau}(t) = \boldsymbol{\tau}^*(t)$, then $\mathbf{s}^*(t_0 - t)$ is a solution of (20) under initial condition $\mathbf{s}(0) = \mathbf{s}^*(t_0)$ and input $\boldsymbol{\tau}(t) = -\boldsymbol{\tau}^*(t_0 - t)$.

The proof of Lemma S.2 is straightforward and stems from standard properties of ordinary differential equations and from the time invariance of (20).

Lemma S.3: Assume that the front-back symmetry property (23) holds. If $\mathbf{s}^*(t)$ is a solution of (20) under initial condition $\mathbf{s}(0) = \mathbf{s}^*(0)$ and input $\boldsymbol{\tau}(t) = \boldsymbol{\tau}^*(t)$, then $\mathbf{M}_s\mathbf{s}^*(t)$ is a solution of (20) under initial condition $\mathbf{s}(0) = \mathbf{M}_s\mathbf{s}^*(0)$ and input $\boldsymbol{\tau}(t) = \mathbf{M}_s\boldsymbol{\tau}^*(t)$.

The proof of this Lemma stems directly from the symmetry relation (23).

Lemma S.4: If $\mathbf{s}^*(t)$ is a solution of (20) under initial condition $\mathbf{s}(0) = \mathbf{s}^*(0)$ and input $\boldsymbol{\tau}(t) = \boldsymbol{\tau}^*(t)$, then $-\mathbf{s}^*(t)$ is a solution of (20) under initial condition $\mathbf{s}(0) = -\mathbf{s}^*(0)$ and input $\boldsymbol{\tau}(t) = -\boldsymbol{\tau}^*(t)$.

The proof of this Lemma stems from the axisymmetry property of (20) given in (22).

Next, Result 1 can be stated and proven. Recall that a T-periodic control input $\boldsymbol{\tau}(t)$ is called *reversible* if it satisfies $\boldsymbol{\tau}(T-t) = -\mathbf{M}_s\boldsymbol{\tau}(t)$. Setting $t' = t - T/2$, this implies that $\boldsymbol{\tau}(t'-T/2) = -\mathbf{M}_s\boldsymbol{\tau}(T/2-t')$.

Result 1: For a front-back symmetric swimmer, i.e. $l_1 = l_2$, if $\boldsymbol{\tau}(t)$ is a reversible input and under initial condition $\mathbf{s}(0) = (\alpha, \alpha)^T$ the solution of (20) satisfies $\mathbf{s}(T/2) = (\beta, \beta)^T$ for some $\alpha, \beta \in I\!R$, then $\mathbf{s}(t)$ is T-periodic.

Proof: First, denote the two half-periods of $\boldsymbol{\tau}(t)$ as $\boldsymbol{\tau}^*(t) = \boldsymbol{\tau}(t)$ for $t \in [0, T/2]$ and $\boldsymbol{\tau}^{**}(t) = \boldsymbol{\tau}(t - T/2)$ for $t \in [T/2, T]$. Similarly, denote $\mathbf{s}^*(t) = \mathbf{s}(t)$ for $t \in [0, T/2]$ and $\mathbf{s}^{**}(t) = \mathbf{s}(t - T/2)$ for $t \in [T/2, T]$. Due to reversibility of $\boldsymbol{\tau}(t)$, it satisfies $\boldsymbol{\tau}^{**}(t) = -\mathbf{M}_s\boldsymbol{\tau}^*(T/2 - t)$. Note that since $\mathbf{s}^*(T/2) = (\beta, \beta)$, the initial condition of $\mathbf{s}^{**}$ satisfies $\mathbf{s}^{**}(0) = \mathbf{s}^*(T/2) = \mathbf{M}_s\mathbf{s}^*(T/2)$. Setting $t_0 = T/2$ and combining Lemma S.2 and S.3 then implies that $\mathbf{s}^{**}(t) = \mathbf{M}_s\mathbf{s}^*(T/2 - t)$. Therefore, one concludes that the the end value of the solution $\mathbf{s}(t)$ satisfies $\mathbf{s}(T) = \mathbf{s}^{**}(T/2) = \mathbf{M}_s\mathbf{s}^*(0) = \mathbf{s}(0)$, hence $\mathbf{s}(t)$ is T-periodic. ■

Recall that a T-periodic control input $\boldsymbol{\tau}(t)$ is called *axisymmetric* if it satisfies $\boldsymbol{\tau}(t+T/2) = -\boldsymbol{\tau}(t)$ for all t. Result 2 is then stated as follows:

Result 2: If $\boldsymbol{\tau}(t)$ is an axisymmetric input and the solution of (20) satisfies $\mathbf{s}(T/2) = -\mathbf{s}(0)$, then $\mathbf{s}(t)$ is T-periodic and satisfies $\mathbf{s}(t + T/2) = -\mathbf{s}(t)$. Moreover, the solution of $\mathbf{q}(t)$ in (19) satisfies $\theta(T) = \theta(0)$.

Proof: The proof of the first part is a direct application of Lemma S.4, combined with shifting the time by $T/2$ due to time invariance of (20). In order to prove the second part concering the orientation angle θ, the following intermediate steps are needed. Denote $\mathbf{G}_\theta(\mathbf{s}) = [1\ 0\ 0]\mathbf{G}(\mathbf{q}, \mathbf{s})$, which is the third row of $\mathbf{G}$ in (19), hence $\theta(t)$ is governed by the differential equation $\dot{\theta} = \mathbf{G}_\theta(\mathbf{s})\dot{\mathbf{s}}$. Note that the structure of $\mathbf{D}(\theta)$ in (24) implies that $[1\ 0\ 0]\mathbf{G}(\mathbf{q}, \mathbf{s}) = [1\ 0\ 0]\mathbf{G}_0(\mathbf{s})$, hence $\mathbf{G}_\theta(\mathbf{s})$ is indeed independent of $\mathbf{q}$. The axisymmetry property (25) implies that $\mathbf{G}_\theta(-\mathbf{s}) = \mathbf{G}_\theta(\mathbf{s})$. Next, recall the definition of $\mathbf{s}^*(t)$ and $\mathbf{s}^{**}(t)$ in the previous proof. It has already been proven that $\mathbf{s}^{**}(t) = -\mathbf{s}^*(t)$. Similarly, denote $\theta^*(t) = \theta(t)$ for $t \in [0, T/2]$ and $\theta^{**}(t) = \theta(t - T/2)$ for $t \in [T/2, T]$, and note that $\theta^{**}(0) =$

$\theta^*(T/2)$. Using these definitions, one obtains

$$\dot{\theta}^{**}(t) = \mathbf{G}_\theta(\mathbf{s}^{**}(t))\dot{\mathbf{s}}^{**}(t) = -\mathbf{G}_\theta(-\mathbf{s}^*(t))\dot{\mathbf{s}}^*(t) = -\mathbf{G}_\theta(\mathbf{s}^*(t))\dot{\mathbf{s}}^*(t) = -\dot{\theta}^*(t).$$

Therefore, it can be concluded that $\theta^{**}(T/2) - \theta^{**}(0) = \theta^*(0) - \theta^*(T/2)$, hence $\theta(0) = \theta(T)$.
■

Result 3: For a front-back symmetric swimmer, i.e. $l_1 = l_2$, if the control input $\boldsymbol{\tau}(t)$ is both reversible and axisymmetric and under initial condition $\mathbf{s}(0) = (\alpha, \alpha)^T$, the solution of (20) satisfies $\mathbf{s}(T/4) = (\beta, -\beta)^T$ for some α, β, then $\mathbf{s}(t)$ is T-periodic and satisfies $\mathbf{s}(T/2) = (-\alpha, -\alpha)^T$ and $\mathbf{s}(3T/4) = (-\beta, \beta)^T$. Moreover, the solution of (19) satisfies $\theta(T) = \theta(0)$ and $y(T) = y(0)$.

Proof: First, we decompose $\boldsymbol{\tau}(t)$ into quarters of period and denote $\boldsymbol{\tau}_i(t) = \boldsymbol{\tau}(t - \frac{i-1}{4}T)$ for $i = 1, 2, 3, 4$ and $t \in [0, T/4]$. Since $\boldsymbol{\tau}(t)$ is both reversible and axisymmetric, it can be verified that its four quarters satisfy the symmetry relations $\boldsymbol{\tau}_2(t) = \mathbf{M}_s\boldsymbol{\tau}_1(T/4 - t)$, $\boldsymbol{\tau}_3(t) = -\boldsymbol{\tau}_1(t)$, and $\boldsymbol{\tau}_4(t) = -\mathbf{M}_s\boldsymbol{\tau}_1(T/4 - t)$. (These relations are illustrated in Figure 2(a).) Next, $\mathbf{s}(t)$ is decomposed similarly into quarters by denoting $\mathbf{s}_i(t) = \mathbf{s}(t - \frac{i-1}{4}T)$ for $i = 1, 2, 3, 4$ and $t \in [0, T/4]$. Note that $\mathbf{s}_2(0) = -\mathbf{M}_s\mathbf{s}_1(T/4)$. Combining Lemmas S.1, S.2, and S.3, one obtains that $\mathbf{s}_2(t) = -\mathbf{M}_s\mathbf{s}_1(T/4 - t)$, hence $\mathbf{s}(T/2) = -\mathbf{M}_s\mathbf{s}(0) = (-\alpha, -\alpha)^T$. Next, either Result 1 or Result 2 implies that $\mathbf{s}(t)$ is T-periodic. Since axisymmetry now implies that $\mathbf{s}_3(t) = -\mathbf{s}_1(t)$ and $\mathbf{s}_4(t) = -\mathbf{s}_2(t)$, one concludes that $\mathbf{s}(3T/4) = (-\beta, \beta)^T$ and $\mathbf{s}(T) = \mathbf{s}(0) = (\alpha, \alpha)^T$. Note that Result 2 also implies that $\theta(T) = \theta(0)$.

Next, the swimmer's state $\mathbf{q}(t)$ is also decomposed as $\mathbf{q}_i(t) = \mathbf{q}(t - \frac{i-1}{4}T)$ for $i = 1, 2, 3, 4$ and $t \in [0, T/4]$, and its components are denoted as $x_i(t)$, $y_i(t)$ and $\theta_i(t)$. The orientation angle $\theta(t)$ satisfies $\dot{\theta} = \mathbf{G}_\theta(\mathbf{s})\dot{\mathbf{s}}$, and recall that the axisymmetry property (25) implied that $\mathbf{G}_\theta(-\mathbf{s}) = \mathbf{G}_\theta(\mathbf{s})$. Moreover, it can also be verified that the front-back symmetry (26) implies that $\mathbf{G}_\theta(\mathbf{M}_s\mathbf{s}) = -\mathbf{G}_\theta(\mathbf{s})\mathbf{M}_s$. Using these symmetries and the relation $\mathbf{s}_2(t) = -\mathbf{M}_s\mathbf{s}_1(T/4 - t)$, one concludes that

$$\begin{aligned}\dot{\theta}_2(t) &= \mathbf{G}_\theta(\mathbf{s}_2(t))\dot{\mathbf{s}}_2(t) = \mathbf{G}_\theta(-\mathbf{M}_s\mathbf{s}_1(T/4 - t))\mathbf{M}_s\dot{\mathbf{s}}_1(T/4 - t)\\ &= -\mathbf{G}_\theta(\mathbf{s}_1(T/4 - t))\mathbf{M}_s^2\dot{\mathbf{s}}_1(T/4 - t) = -\dot{\theta}_1(T/4 - t).\end{aligned}$$

Since $\theta_2(0) = \theta_1(T/4)$, one obtains that $\theta_2(t) = \theta_1(T/4 - t)$, which implies that $\theta(T) = \theta(T/2) = \theta(0)$. That is, the swimmer returns to its initial orientation every half period. Similarly, using the relation $\mathbf{s}_3(t) = -\mathbf{s}_1(t)$, one can obtain that

$$\dot{\theta}_3(t) = \mathbf{G}_\theta(\mathbf{s}_3(t))\dot{\mathbf{s}}_3(t) = -\mathbf{G}_\theta(-\mathbf{s}_1(t))\dot{\mathbf{s}}_1(t) = -\dot{\theta}_1(t).$$

Therefore, one concludes that $\theta_3(t) = -\theta_1(t)$ and, by similar arguments, that $\theta_4(t) = -\theta_2(t) = -\theta_1(T/4 - t)$.

Next, denote $\mathbf{G}_y(\theta, \mathbf{s}) = [0\ 1\ 0]\mathbf{G}(\mathbf{q}, \mathbf{s})$, which is the second row of $\mathbf{G}$ in (19), so that $\dot{y} = \mathbf{G}_y(\theta, \mathbf{s})\dot{\mathbf{s}}$. Using the symmetry relations (24), (25) and (26) it can be verified that $\mathbf{G}_y(-\theta, -\mathbf{s}) = \mathbf{G}_y(\theta, \mathbf{s})$ and that $\mathbf{G}_y(-\theta, \mathbf{M}_s\mathbf{s}) = \mathbf{G}_y(\theta, \mathbf{s})\mathbf{M}_s$. Using all these relations, the time evolution of $y(t)$ then satisfies

$$\begin{aligned}\dot{y}_3(t) &= \mathbf{G}_y(\theta_3(t), \mathbf{s}_3(t))\dot{\mathbf{s}}_3(t) = -\mathbf{G}_y(-\theta_2(T/4 - t), \mathbf{M}_s\mathbf{s}_2(T/4 - t))\mathbf{M}_s\dot{\mathbf{s}}_2(T/4 - t) \\ &= -\mathbf{G}_y(\theta_2(T/4 - t), \mathbf{s}_2(T/4 - t))\mathbf{M}_s^2\dot{\mathbf{s}}_2(T/4 - t) = -\dot{y}_2(T/4 - t).\end{aligned}$$

Since $y_3(0) = y_2(T/4)$, one concludes that $y_3(t) = y_2(T/4 - t)$. By similar arguments, it can be shown that $y_4(t) = y_1(T/4 - t)$. Therefore, $y(t)$ and satisfies $y(T) = y_4(T/4) = y_1(0) = y(0)$, hence $y(t)$ is periodic. Moreover, it also satisfies $y(\frac{3}{4}T) = y(\frac{1}{4}T)$, that is, $y(t)$ returns to its original value once every half period, but with quarter period phase shift. ■

Finally, Result 4 is proven. For a given T-periodic input $\boldsymbol{\tau}(t)$, equation (20) induces a discrete-time dynamical system $\mathbf{s}_{k+1} = \mathcal{H}(\mathbf{s}_k)$ where $\mathbf{s}_k = \mathbf{s}(t = kT)$ and $k \in \{0, 1, 2, 3 \ldots\}$. The map $\mathcal{H}(\cdot)$ is called the Poincaré map of (20).

Result 4: For a front-back symmetric swimmer, i.e. $l_1 = l_2$, if $\boldsymbol{\tau}(t)$ is reversible, then the map $\mathcal{H}$ satisfies $\mathcal{H}(\mathbf{M}_s\mathbf{s}) = \mathbf{M}_s\mathcal{H}^{-1}(\mathbf{s})$.

Proof: First, pick an arbitrary initial condition to (20) as $\mathbf{s}_0 = \mathbf{s}(0)$. Denote by $\mathbf{s}^*(t)$ the solution of (20) for $t \in [0, T]$ under the given input $\boldsymbol{\tau}(t)$ and initial condition $\mathbf{s}_0$, and denote its end value by $\mathbf{s}_T = \mathbf{s}^*(T)$. Using the Poincaré map terminology, $\mathbf{s}_T$ can be equivalently defined as $\mathbf{s}_T = \mathcal{H}(\mathbf{s}_0)$. According to Lemma S.3, $\mathbf{M}_s\mathbf{s}^*(t)$ is the solution of (20) under input $\mathbf{M}_s\boldsymbol{\tau}(t)$ and initial condition $\mathbf{s}(0) = \mathbf{M}_s\mathbf{s}_0$. The reversibility property of the input implies that $\mathbf{M}_s\boldsymbol{\tau}(t) = -\boldsymbol{\tau}(T - t)$. Combining these results with Lemma S.2 then implies that $\mathbf{M}_s\mathbf{s}^*(T - t)$ is the solution of (20) under input $\boldsymbol{\tau}(t)$ and initial condition $\mathbf{s}(0) = \mathbf{M}_s\mathbf{s}_T$. Evaluating at time $t = T$, this relation implies that $\mathcal{H}(\mathbf{M}_s\mathbf{s}_T) = \mathbf{M}_s\mathbf{s}_0$. Using the inverse map $\mathcal{H}^{-1}$ then gives $\mathbf{M}_s\mathbf{s}_T = \mathcal{H}^{-1}(\mathbf{M}_s\mathbf{s}_0)$. Multiplying both sides by $\mathbf{M}_s$ and invoking the original definition $\mathbf{s}_T = \mathcal{H}(\mathbf{s}_0)$ results in $\mathcal{H}(\mathbf{s}_0) = \mathbf{M}_s\mathcal{H}^{-1}(\mathbf{M}_s\mathbf{s}_0)$. Finally, defining $\mathbf{s}' = \mathbf{M}_s\mathbf{s}_0$ gives $\mathcal{H}(\mathbf{M}_s\mathbf{s}') = \mathbf{M}_s\mathcal{H}^{-1}(\mathbf{s}')$, which completes the proof. ■

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