Supplementary document to the paper: "Dynamics of Purcell's three-link microswimmer with a passive elastic tail"

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This supplementary document begins with systematic formulation of the equations of motion for a general articulated swimmer in viscous fluid. Next, explicit formulation for the particular model of Purcell's three-link swimmer under *resistive force theory* (RFT) approximation is given. Finally, details and expressions regarding the model of the swimmer mode with a torsional spring are given.

I. FORCE- AND SHAPE- CONTROLLED ROBOTIC MICROSWIMMER

Consider an articulated robotic swimmer in a viscous fluid. It is assumed that the swimmer is very small, or slow, or the fluid is highly viscous, so that the Reynolds number is nearly zero. This implies that the fluid is governed by *Stokes equations* and inertial effects can be neglected so that the motion is quasistatic [1]. The robot consists of n rigid links which are connected by m joints. The joints are actuated by prescribing either their motion or their internal force or torque. The robot may consist of several open (i.e. serial) kinematic chains, but no closed chains. Let $\mathbf{r}_b$ denote the origin of a body-fixed reference frame which is rigidly attached to the swimmer. Let us attach a moving frame to the ith link, and denote the position of its origin by $\mathbf{r}_i$. The joints, which are enumerated by $\mathcal{J} = \{1 \dots m\}$ are divided into a set of linear joints $\mathcal{J}_{lin} \subseteq \mathcal{J}$ and rotary joints $\mathcal{J}_{rot} \subseteq \mathcal{J}$. A linear joint $j \in \mathcal{J}_{lin}$ imposes linear relative motion between links along the direction of the unit vector $\hat{\mathbf{l}}_j$, with the linear velocity denoted by u_j. A rotary joint $j \in \mathcal{J}_{rot}$ imposes relative rotation between links about an axis whose direction is given by the unit vector $\hat{\mathbf{l}}_j$, where the angular velocity is denoted by u_j and $\mathbf{b}_j$ denotes a point on the axis. The topological structure of the kinematic chains comprising the swimmer is encoded by the indicator matrix I_{ij}, such that $I_{ij} = 1$ if the location of the ith link is affected by the jth joint, and $I_{ij} = 0$ otherwise. Let $\mathbf{v}_i$ and $\boldsymbol{\omega}_i$ denote the linear angular and velocity of the ith link, respectively, and define

$\mathbf{V}_i = \begin{pmatrix} \mathbf{v}_i \\ \boldsymbol{\omega}_i \end{pmatrix}$. Similarly, let $\mathbf{V}_b = \begin{pmatrix} \mathbf{v}_b \\ \boldsymbol{\omega}_b \end{pmatrix}$ denote the linear and angular velocity of the body-fixed reference frame. The kinematic relation between the body velocity, the joints velocities and the links velocities is given by

$$\begin{aligned} \mathbf{v}_i &= \mathbf{v}_b + \boldsymbol{\omega}_b \times (\mathbf{r}_i - \mathbf{r}_b) + \sum_{j \in \mathcal{J}_{lin}} I_{ij} u_j \hat{\mathbf{l}}_j + \sum_{j \in \mathcal{J}_{rot}} I_{ij} u_j \hat{\mathbf{l}}_j \times (\mathbf{r}_i - \mathbf{b}_j) \\ \boldsymbol{\omega}_i &= \boldsymbol{\omega}_b + \sum_{j \in \mathcal{J}_{rot}} I_{ij} u_j \hat{\mathbf{l}}_j. \end{aligned} \tag{1}$$

The kinematic relations (1) can be written in matrix form as

$$\mathbf{V} = \mathbf{T}\mathbf{V}_b + \mathbf{E}\mathbf{u}, \tag{2}$$

where

$$\begin{aligned} &\mathbf{V} = \begin{pmatrix} \mathbf{V}_1 \\ \vdots \\ \mathbf{V}_n \end{pmatrix}, \; \mathbf{T} = \begin{pmatrix} \mathbf{T}_1 \\ \vdots \\ \mathbf{T}_n \end{pmatrix}_{6n \times 6}, \; \mathbf{E} = \begin{pmatrix} \mathbf{E}_{11} & \dots & \mathbf{E}_{1m} \\ \vdots & \vdots & \vdots \\ \mathbf{E}_{n1} & \dots & \mathbf{E}_{nm} \end{pmatrix}_{6n \times m}, \; \mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \\ &\mathbf{T}_i = \begin{pmatrix} \mathbf{I}_{3\times3} & -[(\mathbf{r}_i - \mathbf{r}_b)\times] \\ \mathbf{O}_{3\times3} & \mathbf{I}_{3\times3} \end{pmatrix}, \; \mathbf{E}_{ij} = \begin{cases} I_{ij} \begin{pmatrix} \hat{\mathbf{l}}_j \\ \vec{0} \end{pmatrix} & j \in \mathcal{J}_{lin} \\ I_{ij} \begin{pmatrix} \hat{\mathbf{l}}_j \times (\mathbf{r}_i - \mathbf{b}_j) \\ \hat{\mathbf{l}}_j \end{pmatrix} & j \in \mathcal{J}_{rot} \end{cases} \end{aligned} \tag{3}$$

where $\mathbf{I}$ is the identity matrix and $[\mathbf{a}\times]$ is the cross-product matrix which satisfies $[\mathbf{a}\times]\mathbf{b} = \mathbf{a} \times \mathbf{b}$ for all $\mathbf{a}, \mathbf{b} \in I\!R^3$.

Next, we consider the forces and moments acting on the swimmer's links. Let $\mathbf{f}_i$ and $\mathbf{m}_i$ denote the force and moment exerted by the fluid on the ith link, respectively. Let us also denote $\mathbf{F}_i = \begin{pmatrix} \mathbf{f}_i \\ \mathbf{m}_i \end{pmatrix}$ and $\mathbf{F} = \begin{pmatrix} \mathbf{F}_1 \\ \vdots \\ \mathbf{F}_n \end{pmatrix}$. A fundamental property of viscous flow, which stems from the linearity of Stokes equation, is the existence of a linear relationship between hydrodynamic forces and velocities, which is given by

$$\mathbf{F} = -\boldsymbol{\mathcal{R}}\mathbf{V}. \tag{4}$$

The matrix $\boldsymbol{\mathcal{R}}$ is called the *resistance tensor*, and depends on the positions of all the links. It is shown in [1] that $\boldsymbol{\mathcal{R}}$ is always a symmetric and positive definite matrix. The net force and moment acting on the body-fixed frame, denoted by $\mathbf{f}_b$ and $\mathbf{m}_b$ respectively, are given by

$$\mathbf{f}_b = \sum_{i=1}^{n} \mathbf{f}_i \,, \qquad \mathbf{m}_b = \sum_{i=1}^{n} \mathbf{m}_i + (\mathbf{r}_i - \mathbf{r}_b) \times \mathbf{f}_i. \tag{5}$$

It can be verified that the relation (5) can be written in matrix form as

$$\mathbf{F}_b = \begin{pmatrix} \mathbf{f}_b \\ \mathbf{m}_b \end{pmatrix} = \mathbf{T}^T \mathbf{F}, \tag{6}$$

where the matrix $\mathbf{T}$ defined in (3) is precisely the same as the one in (2). Since the motion in Stokes flow is quasi-static, the forces and torques on the body must vanish $\mathbf{F}_b = 0$. Substituting (4) and (2) into (6) yields

$$\mathbf{F}_b = -\mathbf{T}^T \boldsymbol{\mathcal{R}} (\mathbf{T}\mathbf{V}_b + \mathbf{E}\mathbf{u}) = 0. \tag{7}$$

Equation (7) can be equivalently written as

$$\mathbf{F}_b = -\boldsymbol{\mathcal{R}}_{bb} \mathbf{V}_b - \boldsymbol{\mathcal{R}}_{bu} \mathbf{u} = 0, \text{ where } \boldsymbol{\mathcal{R}}_{bb} = \mathbf{T}^T \boldsymbol{\mathcal{R}} \mathbf{T} \text{ and } \boldsymbol{\mathcal{R}}_{bu} = \mathbf{T}^T \boldsymbol{\mathcal{R}} \mathbf{E}. \tag{8}$$

The interpretation of (8) is as follows. The matrix $\boldsymbol{\mathcal{R}}_{bb}$ gives the relation between force and moment acting on the body and its rigid-body velocity, assuming that all joints are locked (i.e. "dragging resistance" [2]). The matrix $\boldsymbol{\mathcal{R}}_{bu}$ gives the relation between force and moment acting on the body and the joints velocities, assuming that the body frame is held fixed (i.e. "pumping resistance" [2]). Inverting the relation (8) then gives a linear relation between the rigid-body velocity and the joints velocities of the swimmer, as

$$\mathbf{V}_b = -\boldsymbol{\mathcal{R}}_{bb}^{-1} \boldsymbol{\mathcal{R}}_{bu} \mathbf{u}. \tag{9}$$

Assuming that the joints velocities $\mathbf{u}$ are directly controlled by the robot, Equation (9) gives the resulting swimmer's body velocity $\mathbf{V}_b$ for any prescribed value of $\mathbf{u}$.

Force-controlled swimmer: Another important possible scenario is when the robot does not directly control its joints velocities $\mathbf{u}$, but instead it prescribes the *internal forces and moments* supplied at the actuated joints. In order to study this case, we need to consider additional force-and-moment balance on a section of the swimmer, which is cut at the jth joint, as follows. The partial kinematic chain composed of the links with indices $\mathcal{I}_j = \{i : I_{ij} = 1\}$ is subjected to force and moment exerted by the rest of the robot through the jth joint, which are denoted by $\tilde{\mathbf{f}}^j$ and $\tilde{\mathbf{m}}^j$, respectively. Due to static equilibrium of the partial kinematic chain, these force and moment are balanced by the hydrodynamic forces and moments $\mathbf{f}_i$ and $\mathbf{m}_i$ for $i \in \mathcal{I}_j$, giving rise to the following relations:

$$\tilde{\mathbf{f}}^j = -\sum_{i \in \mathcal{I}_j} \mathbf{f}_i \quad , \quad \tilde{\mathbf{m}}^j = -\sum_{i \in \mathcal{I}_j} (\mathbf{m}_i + (\mathbf{r}_i - \mathbf{b}_j) \times \mathbf{f}_i) \ , j = 1 \ldots m. \tag{10}$$

The controlled force or moment at each joint, denoted by τ_j, is directed along the joint axis, and is given by

$$\tau_j = \begin{cases} \hat{\mathbf{l}}_j \cdot \tilde{\mathbf{f}}^j & j \in \mathcal{J}_{lin} \\ \hat{\mathbf{l}}_j \cdot \tilde{\mathbf{m}}^j & j \in \mathcal{J}_{rot} \end{cases} . \tag{11}$$

Denoting $\boldsymbol{\tau} = (\tau_1 \ldots \tau_m)^T$ as the vector of controlled forces or moments at the actuated joints, it can be verified that the projection of the balance equation (10) along the relevant direction for the jth joint according to (11) gives rise to the matrix-form equation

$$\boldsymbol{\tau} = -\mathbf{E}^T \mathbf{F}, \tag{12}$$

where the matrix $\mathbf{E}$ defined in (3) is precisely the same matrix used in the kinematic relation (2). The fact that the matrices $\mathbf{T}^T$ and $\mathbf{E}^T$ in the force-and-moment balances (6) and (12) are transpose of the matrices in the kinematic relation (2) is a standard result in formulation of kinematics and statics of serial robots which is a consequence of the principle of virtual work, cf. [3, 4], where $\mathbf{T}$ and $\mathbf{E}$ can be interpreted as the robot's Jacobian matrices. Next, substituting (2), (4) and (9) into (12) gives

$$\boldsymbol{\tau} = \mathbf{E}^T \boldsymbol{\mathcal{R}} \mathbf{V} = \mathbf{E}^T \boldsymbol{\mathcal{R}} (\mathbf{T}\mathbf{V}_b + \mathbf{E}\mathbf{u}) = \left(\mathbf{E}^T \boldsymbol{\mathcal{R}} \mathbf{E} - \mathbf{E}^T \boldsymbol{\mathcal{R}} \mathbf{T} (\mathbf{T}^T \boldsymbol{\mathcal{R}} \mathbf{T})^{-1} \mathbf{T}^T \boldsymbol{\mathcal{R}} \mathbf{E}\right) \mathbf{u}. \tag{13}$$

Denoting $\boldsymbol{\mathcal{R}}_{uu} = \mathbf{E}^T \boldsymbol{\mathcal{R}} \mathbf{E}$, this equation can be written as

$$\boldsymbol{\tau} = \left(\boldsymbol{\mathcal{R}}_{uu} - \boldsymbol{\mathcal{R}}_{bu}^T \boldsymbol{\mathcal{R}}_{bb}^{-1} \boldsymbol{\mathcal{R}}_{bu}\right) \mathbf{u}, \tag{14}$$

where $\boldsymbol{\mathcal{R}}_{bb}$ and $\boldsymbol{\mathcal{R}}_{bu}$ are defined in (8). $\boldsymbol{\mathcal{R}}_{uu}$ can be interpreted as the forces/moments exerted on the joints axes by the fluid (i.e. $-\boldsymbol{\tau}$) under given joints velocities $\mathbf{u}$, assuming that the body frame is held fixed. Equation (14) can then be inverted in order to obtain $\mathbf{u}$ as a function of $\boldsymbol{\tau}$ as

$$\mathbf{u} = \mathbf{H}\boldsymbol{\tau}, \text{ where } \mathbf{H} = \left(\boldsymbol{\mathcal{R}}_{uu} - \boldsymbol{\mathcal{R}}_{bu}^T \boldsymbol{\mathcal{R}}_{bb}^{-1} \boldsymbol{\mathcal{R}}_{bu}\right)^{-1}. \tag{15}$$

Note that by definition, the matrix $\mathbf{H}$ in (15) is symmetric. Finally, substituting the expression for $\mathbf{u}$ from (14) into (9) gives

$$\mathbf{V}_b = -\boldsymbol{\mathcal{R}}_{bb}^{-1}\boldsymbol{\mathcal{R}}_{bu}\left(\boldsymbol{\mathcal{R}}_{uu} - \boldsymbol{\mathcal{R}}_{bu}^T\boldsymbol{\mathcal{R}}_{bb}^{-1}\boldsymbol{\mathcal{R}}_{bu}\right)^{-1}\boldsymbol{\tau}. \tag{16}$$

Possible reductions of the general formulation above can be obtained in some simplified cases, as follows. First, consider the case of a planar swimmer which translates in xy plane while all rotations are along the z-axis. In this case, one can reduce the linear velocities $\mathbf{v}_i, \mathbf{v}_b$ and linear forces $\mathbf{f}_i, \mathbf{f}_b$ to their (x, y)-components and the angular velocities $\boldsymbol{\omega}_i, \boldsymbol{\omega}_b$ and moments $\mathbf{m}_i, \mathbf{m}_b$ to their z-component, leading to twofold reduction in the dimension of the resistance matrix $\boldsymbol{\mathcal{R}}$ to $3n \times 3n$. The matrices $\mathbf{T}$ and $\mathbf{E}$ in (3) reduce to dimensions $3n \times 3$ and $3n \times m$, respectively. The matrix $[(\mathbf{r}_i - \mathbf{r}_b)\times]$ in $\mathbf{T}$ is replaced with the row vector $(\bar{\mathbf{r}}_i - \bar{\mathbf{r}}_b)^T\mathbf{J}^T$ and the column vector $\hat{\mathbf{l}}_j \times (\mathbf{r}_i - \mathbf{b}_j) = \hat{\mathbf{z}} \times (\mathbf{r}_i - \mathbf{b}_j)$ is replaced with $\mathbf{J}(\bar{\mathbf{r}}_i - \bar{\mathbf{b}}_j)$, where $\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\bar{\mathbf{r}}_i, \bar{\mathbf{r}}_b, \bar{\mathbf{b}}_j$ are the (x, y)-components of $\mathbf{r}_i, \mathbf{r}_b, \mathbf{b}_j$, respectively.

Second, consider the case where the hydrodynamic interaction between the rigid links is assumed negligible, such as, for example, when the swimmer consists of slender links [5]. In this case, the resistance tensor becomes block-diagonal $\boldsymbol{\mathcal{R}} = \mathrm{diag}(\boldsymbol{\mathcal{R}}_1, \ldots, \boldsymbol{\mathcal{R}}_n)$, so that the force-velocity relation for the ith link is $\mathbf{F}_i = -\boldsymbol{\mathcal{R}}_i\mathbf{V}_i$. Thus, the resistance matrices $\boldsymbol{\mathcal{R}}_{bb}, \boldsymbol{\mathcal{R}}_{bu}$ and $\boldsymbol{\mathcal{R}}_{uu}$ defined in (8) and (16) reduce to

$$\boldsymbol{\mathcal{R}}_{bb} = \sum_{i=1}^{n} \mathbf{T}_i^T\boldsymbol{\mathcal{R}}_i\mathbf{T}_i\ ,\ \boldsymbol{\mathcal{R}}_{bu} = \sum_{i=1}^{n} \mathbf{T}_i^T\boldsymbol{\mathcal{R}}_i\mathbf{E}_i \text{ and } \boldsymbol{\mathcal{R}}_{uu} = \sum_{i=1}^{n} \mathbf{E}_i^T\boldsymbol{\mathcal{R}}_i\mathbf{E}_i, \text{ where } \mathbf{E}_i = (\mathbf{E}_{i1} \ldots \mathbf{E}_{im}). \tag{17}$$

Mechanical power expenditure: The mechanical power (i.e. rate of work) dissipated by the fluid's viscous drag forces and torques resisting the motion of all links is $-\mathbf{F} \cdot \mathbf{V} = \mathbf{V}^T\boldsymbol{\mathcal{R}}\mathbf{V} > 0$. The mechanical power expended by the controlled internal forces and torques is given by $\boldsymbol{\tau} \cdot \mathbf{u} = \mathbf{u}\mathbf{H}^{-1}\mathbf{u}$. Not surprisingly, using the definitions of $\mathbf{H}$, $\boldsymbol{\mathcal{R}}_{bb}$, $\boldsymbol{\mathcal{R}}_{bu}$ and $\boldsymbol{\mathcal{R}}_{uu}$, straightforward manipulation can show that $\mathbf{V}^T\boldsymbol{\mathcal{R}}\mathbf{V} = \mathbf{u}\mathbf{H}^{-1}\mathbf{u}$. This also implies that the matrix $\mathbf{H}$ defined in (15) is positive definite.

II. EQUATIONS OF MOTION OF PURCELL'S THREE-LINK SWIMMER

A sketch of Purcell's three swimmer model appears in Figure 1(a). It consists of three rigid links of lengths l_0, l_1, l_2 and two revolute joints actuated by internal torques. The shape of the swimmer is described by the two relative angles between the links, denoted by $\mathbf{\Phi}=(\phi_1,\phi_2)^T$. The state of the swimmer is described by the planar position and orientation of a reference frame which is attached to the central link, and is denoted by $\mathbf{q}=(x,y,\theta)^T$. It is assumed that the swimmer is submerged in an unbounded fluid domain of viscosity μ, and that is motion is confined to the xy plane.

In order to obtain an explicit formulation of the swimmer's dynamic equation of motions, the simplest available model of *resistive force theory* (RFT) for slender bodies [5, 6] will be used. This theory approximates the hydrodynamic forces and torques exerted by a viscous fluid on a moving slender body, as follows. Consider a slender filament of length l and circular cross-section of radius a. Let $\mathbf{v}(s)$ denote the instantaneous local velocity of the filament at a location s along its axis. This velocity can be decomposed into its axial and normal components as $\mathbf{v}(s) = \mathbf{v}_t(s) + \mathbf{v}_n(s)$, where $\mathbf{v}_t(s) = (\mathbf{e}_t(s)\cdot\mathbf{v}(s))\mathbf{e}_t(s)$ and $\mathbf{e}_t(s)$ is a unit vector tangent to the axis of the filament at location s. The basic result of slender body theory states that to leading order in $\ln(l/a)$, the local force acting at location s is given by $\mathbf{f}(s) = -c_n\mathbf{v}_n(s) - c_t\mathbf{v}_t(s)$ where $c_t = 2\pi\mu/\ln(l/a)$ and $c_n = 2c_t$. Next, it is assumed that the filament is a straight rigid rod whose motion is confined to a plane. Let $\mathbf{v}$ and ω denote the linear and angular velocities of a reference frame located at the center of the rod, and let $\mathbf{e}_t$ denote be a unit vector along the rod's axis. It is then straightforward to show (cf. [7]) that the net force and torque acting on the rod is $\mathbf{f} = -c_t l\mathbf{v}_t - c_n l\mathbf{v}_n$ and $\tau = -c_n l^3/12\omega$, where $\mathbf{v}_t = (\mathbf{v}\cdot\mathbf{e}_t)\mathbf{e}_t$ and $\mathbf{v}_n = \mathbf{v} - \mathbf{v}_t$, and τ is taken with respect to the center of the rod. The swimmer consists of three links, labeled $0, 1$ and 2, and it is assumed that the three links do not interact hydrodynamically, which is correct to leading order for slender bodies. Using the notation defined in the previous section, the 3×3 resistance matrices $\boldsymbol{\mathcal{R}}_i$ for planar motion of the three links are given by

$$\boldsymbol{\mathcal{R}}_i = c_t l_i \begin{pmatrix} 1+\sin^2(\alpha_i) & -\cos(\alpha_i)\sin(\alpha_i) & 0 \\ -\cos(\alpha_i)\sin(\alpha_i) & 1+\cos^2(\alpha_i) & 0 \\ 0 & 0 & l_i^2/6 \end{pmatrix} \quad (18)$$

$$\text{where } \alpha_0=\theta,\ \alpha_1=\theta+\phi_1,\ \text{and } \alpha_2=\theta-\phi_2.$$

The matrices $\mathbf{T}_i$ and $\mathbf{E}_i$, which encode the geometric structure of the swimmer and its

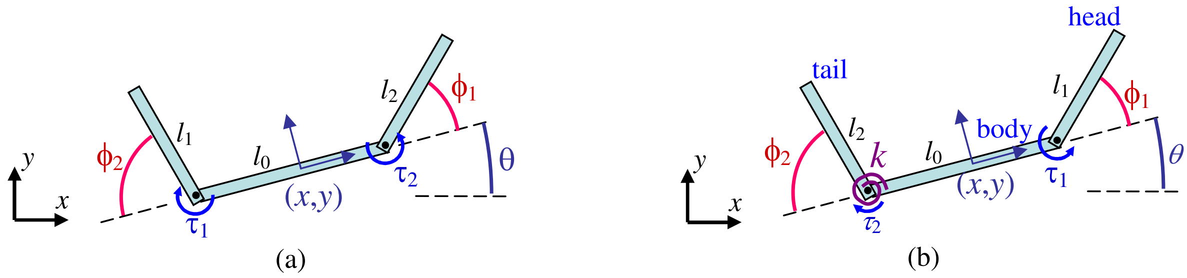

FIG. 1: (a) Purcell's three-link swimmer model. (b) The swimmer model with a torsional spring.

actuated joints, are given by

$$\mathbf{T}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{T}_1 = \begin{pmatrix} 1 & 0 & -\frac{1}{2}(l_0 \sin\theta + l_1 \sin\alpha_1) \\ 0 & 1 & \frac{1}{2}(l_0 \cos\theta + l_1 \cos\alpha_1) \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{T}_2 = \begin{pmatrix} 1 & 0 & \frac{1}{2}(l_0 \sin\theta + l_2 \sin\alpha_2) \\ 0 & 1 & -\frac{1}{2}(l_0 \cos\theta + l_2 \cos\alpha_2) \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{E}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{E}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \mathbf{E}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix} \tag{19}$$

where α_1 and α_2 are defined in (18). Using (18) and (19), the expressions for $\boldsymbol{\mathcal{R}}_{bb}$, $\boldsymbol{\mathcal{R}}_{bu}$ and $\boldsymbol{\mathcal{R}}_{uu}$ can be formulated according to (17), and equations (14) and (16) which govern the swimmer's dynamics can be formulated. Note that $\mathbf{u}$ is precisely the vector of velocities of shape variables, $\mathbf{u} = \dot{\mathbf{\Phi}}$. Thus, equations (14) and (16) can be written in the form

$$\dot{\mathbf{q}} = \mathcal{G}(\mathbf{q}, \mathbf{\Phi})\dot{\mathbf{\Phi}}. \tag{20}$$

$$\dot{\mathbf{\Phi}} = \mathbf{H}(\mathbf{\Phi})\boldsymbol{\tau}. \tag{21}$$

Note that $\mathbf{H}(\mathbf{\Phi})$ in (21) is precisely $\mathbf{H}$ defined in (15). It can be shown that $\mathbf{H}$ depends only on the shape variables $\mathbf{\Phi}$ and not on the postion of the swimmer $\mathbf{q}$, since the fluid domain is unbounded. Additionally, the swimmer's velocity $\dot{\mathbf{q}}$ expressed in body-fixed reference frame is also independent of $\mathbf{q}$ and depends only on the shape variables $\mathbf{\Phi}$. Thus, Eq. (20) can be rewritten as

$$\dot{\mathbf{q}} = \mathbf{D}(\theta)\mathbf{G}(\mathbf{\Phi})\dot{\mathbf{\Phi}}, \text{ where } \mathbf{D}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \mathbf{G}(\mathbf{\Phi}) = \mathcal{G}(0, \mathbf{\Phi}). \tag{22}$$

Equations (22) and (21) are precisely equations (1) and (2) from the submitted paper.

Next, we give explicit expressions for equations (22) and (21) under the RFT formulation. As a preliminary step, we define the transformation $\Upsilon[f(l_1, l_2, \phi_1, \phi_2)] = f(l_2, l_1, \phi_2, \phi_1)$, which is simply a re-evaluation of the expression in $f(\cdot)$ after interchanging between the two side links. The expressions for elements of the symmetric matrix $\mathbf{W}(\mathbf{\Phi}) = \mathbf{H}^{-1}(\mathbf{\Phi})$ are then given as

$$
\begin{aligned}
W_{ij} &= \frac{w_{ij}}{\Delta_W} \text{ where}\\
w_{11} &= -l_1^3 c_t (l_0^3 l_1 l_2 \cos(2\phi_1+2\phi_2) - 16 l_0 l_1 l_2^3 \cos(\phi_1)^2 - 48 l_0 l_1^2 l_2^2 \cos(\phi_2) - 144 l_0^2 l_1 l_2^2 \cos(\phi_2) + 36 l_0^3 l_1 l_2 \cos(\phi_1)^2 - 36 l_0^3 l_1 l_2 \cos(\phi_2)^2\\
&\quad - 3 l_0^3 l_1 l_2 \cos(2\phi_1-2\phi_2) + 12 l_0^2 l_1 l_2^2 \cos(2\phi_1+\phi_2) + 48 l_0^2 l_1 l_2^2 \cos(\phi_1)^2 - 24 l_0^2 l_1^2 l_2 \cos(\phi_2)^2 - 12 l_0^2 l_1 l_2^2 \cos(2\phi_1-\phi_2) - 72 l_0 l_1 l_2^3 \cos(\phi_2)\\
&\quad + 24 l_0 l_1 l_2^3 \cos(2\phi_1+\phi_2) - 40 l_0^4 l_1 - 64 l_0 l_2^4 - 64 l_0^4 l_2 - 36 l_1 l_2^4 - 16 l_0^3 l_1^2 - 64 l_0^2 l_2^3 - 64 l_0^3 l_2^2 - 16 l_1^2 l_2^3 - 64 l_0 l_1 l_2^3 - 96 l_0^3 l_2^2 \cos(\phi_2)\\
&\quad - 16 l_0 l_2^4 \cos(\phi_2)^2 - 96 l_0^2 l_2^3 \cos(\phi_2) + 8 l_0^4 l_1 \cos(\phi_1)^2 - 126 l_0^3 l_1 l_2 - 16 l_0^4 l_2 \cos(\phi_2)^2 + 4 l_1 l_2^4 \cos(2\phi_1+2\phi_2) - 24 l_0^2 l_1^2 l_2 - 96 l_0^2 l_1 l_2^2 - 16 l_0^5 - 16 l_2^5)\\[1ex]
w_{12} &= 2 l_1^2 l_2^2 c_t (18 l_0 l_1 l_2^2 \cos(\phi_1) - 6 l_0 l_1 l_2^2 \cos(\phi_1+2\phi_2) + 18 l_1^2 l_2^2 - 2 l_1^2 l_2^2 \cos(2\phi_1+2\phi_2) + 32 l_0^2 l_1 l_2 + 32 l_0 l_1 l_2^2 + 8 l_1 l_2^3 + 8 l_1^3 l_2 + 24 l_0^3 l_2 \cos(\phi_1)\\
&\quad + 12 l_0 l_2^3 \cos(\phi_1) + 24 l_0^3 l_1 \cos(\phi_2) + 12 l_0 l_1^3 \cos(\phi_2) + 3 l_0^2 l_1^2 \cos(2\phi_1-\phi_2) - 6 l_0^3 l_1 \cos(\phi_1+\phi_2) - 6 l_0^3 l_2 \cos(\phi_1+\phi_2) + 32 l_0 l_1^2 l_2\\
&\quad - 3 l_0^2 l_1^2 \cos(2\phi_1+\phi_2) + 18 l_0^3 l_1 \cos(\phi_1-\phi_2) + 18 l_0^3 l_2 \cos(\phi_1-\phi_2) + 36 l_0^2 l_1^2 \cos(\phi_2) + 36 l_0^2 l_2^2 \cos(\phi_1) - 3 l_0^2 l_2^2 \cos(\phi_1+2\phi_2)\\
&\quad + 3 l_0^2 l_2^2 \cos(\phi_1-2\phi_2) + 12 l_0^4 \cos(\phi_1-\phi_2) + 8 l_0 l_1^2 l_2 \cos(\phi_1)^2 + 8 l_0 l_1 l_2^2 \cos(\phi_2)^2 - 18 l_0^2 l_1 l_2 \cos(\phi_1+\phi_2) + 24 l_0^2 l_1 l_2 \cos(\phi_1)\\
&\quad + 18 l_0 l_1^2 l_2 \cos(\phi_2) + 24 l_0^2 l_1 l_2 \cos(\phi_2) + 18 l_0^2 l_1 l_2 \cos(\phi_1-\phi_2) - 6 l_0 l_1^2 l_2 \cos(2\phi_1+\phi_2))\\[1ex]
\Delta_W &= 3 l_0^3 l_1 l_2 \cos(2\phi_1+2\phi_2) - 48 l_0 l_1 l_2^3 \cos(\phi_1)^2 + 288 l_0 l_1^2 l_2^2 \cos(\phi_2) + 432 l_0^2 l_1 l_2^2 \cos(\phi_2) + 108 l_0^3 l_1 l_2 \cos(\phi_1)^2 + 108 l_0^3 l_1 l_2 \cos(\phi_2)^2\\
&\quad - 9 l_0^3 l_1 l_2 \cos(2\phi_1-2\phi_2) + 36 l_0^2 l_1 l_2^2 \cos(2\phi_1+\phi_2) + 144 l_0^2 l_1 l_2^2 \cos(\phi_1)^2 + 144 l_0^2 l_1^2 l_2 \cos(\phi_2)^2 - 36 l_0^2 l_1 l_2^2 \cos(2\phi_1-\phi_2) + 216 l_0 l_1 l_2^3 \cos(\phi_2)\\
&\quad + 72 l_0 l_1 l_2^3 \cos(2\phi_1+\phi_2) + 288 l_0 l_1^2 l_2^2 \cos(\phi_1) + 432 l_0^2 l_1^2 l_2 \cos(\phi_1) - 48 l_0 l_1^3 l_2 \cos(\phi_2)^2 - 72 l_0 l_1^2 l_2^2 \cos(\phi_1-\phi_2) - 36 l_0^2 l_1^2 l_2 \cos(\phi_1-2\phi_2)\\
&\quad + 36 l_0^2 l_1^2 l_2 \cos(\phi_1+2\phi_2) + 216 l_0 l_1^3 l_2 \cos(\phi_1) + 72 l_0 l_1^3 l_2 \cos(\phi_1+2\phi_2) + 216 l_0 l_1^2 l_2^2 \cos(\phi_1+\phi_2) + 96 l_0^4 l_1 + 96 l_0 l_2^4 + 96 l_0^4 l_2 + 108 l_1 l_2^4\\
&\quad + 96 l_0^3 l_1^2 + 96 l_0^2 l_2^3 + 96 l_0^3 l_2^2 + 96 l_1^2 l_2^3 + 96 l_0 l_1^4 + 108 l_1^4 l_2 + 96 l_0^2 l_1^3 + 96 l_1^3 l_2^2 + 144 l_0^3 l_1^2 \cos(\phi_1) + 24 l_0 l_1^4 \cos(\phi_1)^2 + 240 l_0 l_1^3 l_2\\
&\quad + 144 l_1^3 l_2^2 \cos(\phi_1+\phi_2) + 240 l_0 l_1 l_2^3 + 144 l_0^3 l_2^2 \cos(\phi_2) + 144 l_1^2 l_2^3 \cos(\phi_1+\phi_2) + 12 l_1^4 l_2 \cos(2\phi_1+2\phi_2) + 24 l_0 l_2^4 \cos(\phi_2)^2 + 144 l_0^2 l_2^3 \cos(\phi_2)\\
&\quad + 144 l_0^2 l_1^3 \cos(\phi_1) + 24 l_0^4 l_1 \cos(\phi_1)^2 + 270 l_0^3 l_1 l_2 + 24 l_0^4 l_2 \cos(\phi_2)^2 + 12 l_1 l_2^4 \cos(2\phi_1+2\phi_2) + 144 l_0^2 l_1^2 l_2 + 144 l_0^2 l_1 l_2^2 + 24 l_0^5 + 24 l_2^5 + 24 l_1^5\\[1ex]
w_{21} &= w_{12} \text{ , and } w_{22} = \Upsilon[w_{11}].
\end{aligned}
\tag{23}
$$

The explicit expressions for $\mathbf{G}(\mathbf{\Phi})$ are given as

$$G_{ij} = \frac{g_{ij}}{\Delta_G} \text{ where}$$

$$\begin{aligned}
g_{11} = {} & 2l_1^2(6l_0l_2^3\sin(\phi_1-\phi_2)+18l_0^2l_2^2\sin(\phi_1-\phi_2)-6l_1^2l_2^2\sin(\phi_2)+16l_0l_2^3\sin(\phi_1)+8l_0^3l_1\sin(\phi_1)+6l_0^2l_2^2\sin(\phi_1+\phi_2)-6l_0l_2^3\sin(\phi_1+\phi_2)\\
& +12l_0^3l_2\sin(\phi_1)+2l_0^2l_1^2\sin(2\phi_1)-8l_1l_2^3\sin(\phi_2)+l_0^3l_2\sin(\phi_1+2\phi_2)+8l_1l_2^3\sin(\phi_1)+3l_0^3l_2\sin(\phi_1-2\phi_2)+2l_1^2l_2^2\sin(2\phi_1+\phi_2)\\
& -2l_2^4\sin(\phi_1+2\phi_2)+4l_0^4\sin(\phi_1)+6l_2^4\sin(\phi_1)+3l_0l_1^2l_2\sin(2\phi_1)-3l_0l_1^2l_2\sin(2\phi_2)+18l_0^2l_1l_2\sin(\phi_1)-16l_0l_1l_2^2\sin(\phi_2)\\
& +l_0l_1^2l_2\sin(2\phi_1+2\phi_2)+12l_0l_1l_2^2\sin(\phi_1-\phi_2)+3l_0^2l_1l_2\sin(\phi_1-2\phi_2)+3l_0^2l_1l_2\sin(\phi_1+2\phi_2)-4l_0^2l_1l_2\sin(2\phi_2)+12l_0l_1l_2^2\sin(\phi_1+\phi_2))
\end{aligned}$$

$$\begin{aligned}
g_{21} = {} & -l_1^2(-6l_0^2l_1^2+4l_0^3l_1\cos(\phi_1)+12l_0^2l_1l_2+16l_0l_1l_2^2-4l_0l_1^3+12l_1^2l_2^2\cos(\phi_2)+16l_1l_2^3\cos(\phi_2)+24l_0^3l_2\cos(\phi_1)+16l_0l_2^3\cos(\phi_1)+24l_0^2l_2^2\cos(\phi_1)\\
& +18l_0^2l_1l_2\cos(\phi_1)+6l_0l_1^2l_2\cos(2\phi_1)+2l_0^2l_1^2\cos(2\phi_1)+l_0^3l_2\cos(\phi_1+2\phi_2)+12l_0l_2^3\cos(\phi_1-\phi_2)+24l_0l_2^3\cos(\phi_1+\phi_2)+18l_0^2l_2^2\cos(\phi_1\\
& +\phi_2)+3l_0^3l_2\cos(\phi_1-2\phi_2)+4l_1^2l_2^2\cos(2\phi_1+\phi_2)+6l_0l_1^2l_2\cos(2\phi_2)+16l_0l_1l_2^2\cos(\phi_2)+24l_0l_1l_2^2\cos(\phi_1-\phi_2)+6l_0^2l_1l_2\cos(\phi_1-2\phi_2)\\
& +4l_0^2l_1l_2\cos(2\phi_2)+12l_0l_1l_2^2\cos(\phi_1+\phi_2)+18l_0^2l_2^2\cos(\phi_1-\phi_2)+16l_1l_2^3\cos(\phi_1)+12l_2^4\cos(\phi_1)+4l_0^4\cos(\phi_1)+4l_2^4\cos(\phi_1+2\phi_2))
\end{aligned}$$

$$\begin{aligned}
g_{31} = {} & -2l_1^2(16l_0l_1^2+16l_0^2l_1+16l_1l_2^2+18l_1^2l_2+4l_1^3+12l_2^3\cos(\phi_1+\phi_2)+12l_0^3\cos(\phi_1)+18l_0l_2^2\cos(\phi_1+\phi_2)+12l_1l_2^2\cos(\phi_1+\phi_2)\\
& +12l_0^2l_1\cos(\phi_1)+24l_0l_2^2\cos(\phi_1)+36l_0^2l_2\cos(\phi_1)+40l_0l_1l_2-6l_0l_2^2\cos(\phi_1-\phi_2)-3l_0^2l_2\cos(\phi_1-2\phi_2)+3l_0^2l_2\cos(\phi_1+2\phi_2)\\
& +4l_0l_1^2\cos(\phi_1)^2+2l_1^2l_2\cos(2\phi_1+2\phi_2)+18l_0l_1l_2\cos(\phi_1)+6l_0l_1l_2\cos(\phi_1+2\phi_2)-8l_0l_1l_2\cos(\phi_2)^2)
\end{aligned}$$

$$\begin{aligned}
\Delta_G = {} & l_0^3l_1l_2\cos(2\phi_1+2\phi_2)-16l_0l_1l_2^3\cos(\phi_1)^2+96l_0l_1^2l_2^2\cos(\phi_2)+144l_0^2l_1l_2^2\cos(\phi_2)+36l_0^3l_1l_2\cos(\phi_1)^2+36l_0^3l_1l_2\cos(\phi_2)^2\\
& -3l_0^3l_1l_2\cos(2\phi_1-2\phi_2)+12l_0^2l_1l_2^2\cos(2\phi_1+\phi_2)+48l_0^2l_1l_2^2\cos(\phi_1)^2+48l_0^2l_1^2l_2\cos(\phi_2)^2-12l_0^2l_1l_2^2\cos(2\phi_1-\phi_2)+72l_0l_1l_2^3\cos(\phi_2)\\
& +24l_0l_1l_2^3\cos(2\phi_1+\phi_2)+96l_0l_1^2l_2^2\cos(\phi_1)+144l_0^2l_1^2l_2\cos(\phi_1)-16l_0l_1^3l_2\cos(\phi_2)^2-24l_0l_1^2l_2^2\cos(\phi_1-\phi_2)-12l_0^2l_1^2l_2\cos(\phi_1-2\phi_2)\\
& +12l_0^2l_1^2l_2\cos(\phi_1+2\phi_2)+72l_0l_1^3l_2\cos(\phi_1)+24l_0l_1^3l_2\cos(\phi_1+2\phi_2)+72l_0l_1^2l_2^2\cos(\phi_1+\phi_2)+32l_0^4l_1+32l_0l_2^4+32l_0^4l_2+36l_1l_2^4\\
& +32l_0^3l_1^2+32l_0^2l_2^3+32l_0^3l_2^2+32l_1^2l_2^3+32l_0l_1^4+36l_1^4l_2+32l_0^2l_1^3+32l_1^3l_2^2+48l_0^3l_1^2\cos(\phi_1)+8l_0l_1^4\cos(\phi_1)^2+80l_0l_1^3l_2+48l_1^3l_2^2\cos(\phi_1+\phi_2)\\
& +80l_0l_1l_2^3+48l_0^3l_2^2\cos(\phi_2)+48l_1^2l_2^3\cos(\phi_1+\phi_2)+4l_1^4l_2\cos(2\phi_1+2\phi_2)+8l_0l_2^4\cos(\phi_2)^2+48l_0^2l_2^3\cos(\phi_2)+48l_0^2l_1^3\cos(\phi_1)\\
& +8l_0^4l_1\cos(\phi_1)^2+90l_0^3l_1l_2+8l_0^4l_2\cos(\phi_2)^2+4l_1l_2^4\cos(2\phi_1+2\phi_2)+48l_0^2l_1^2l_2+48l_0^2l_1l_2^2+8l_0^5+8l_2^5+8l_1^5
\end{aligned}$$

$$g_{12} = -\Upsilon[g_{11}],\; g_{22} = \Upsilon(g_{21}), \text{ and } g_{32} = -\Upsilon(g_{31}). \tag{24}$$

III. THE THREE-LINK SWIMMER WITH A TORSIONAL SPRING:

We now formulate the equations of motion for the three-link swimmer where the second joint is actuated by a torsional spring generating torque according to $\tau_2 = -k\phi_2$, which represents an elastic tail. In order to formulate the governing dynamic equation for the swimmer with a torsional spring, further manipulation of (22) and (21) is needed, as follows. Inverting (21), one obtains $\tau_2 = (-H_{21}(\mathbf{\Phi})\dot{\phi}_1 + H_{11}(\mathbf{\Phi})\dot{\phi}_2)/\Delta_H(\mathbf{\Phi})$, where

$\Delta_H(\mathbf{\Phi}) = \det(\mathbf{H}(\mathbf{\Phi})) = H_{11}H_{22} - H_{21}H_{12}$ and H_{ij} is the (i,j)-element of $\mathbf{H}(\mathbf{\Phi})$. Substitution of the elasticity relation $\tau_2 = -k\phi_2$ then gives

$$\dot{\phi}_2 = -kN(\mathbf{\Phi})\phi_2 - F(\mathbf{\Phi})\dot{\phi}_1, \tag{25}$$

where $N(\mathbf{\Phi}) = \Delta_H(\mathbf{\Phi})/H_{11}(\mathbf{\Phi})$ and $F(\mathbf{\Phi}) = -H_{21}(\mathbf{\Phi})/H_{11}(\mathbf{\Phi})$. The explicit expressions for $N(\mathbf{\Phi})$ and $F(\mathbf{\Phi})$ under the RFT formulation are too long to show here. Additional expressions evaluated at $\mathbf{\Phi} = 0$ are defined in the submitted paper, and are used to formulate the leading order dynamic solution. These expressions under the RFT formulation are given as follows:

$$\begin{aligned}
N_0 &= N(\mathbf{\Phi}=0) = \frac{\Delta_H(\mathbf{\Phi}=0)}{H_{11}(\mathbf{\Phi}=0)} = \frac{3(l_0+l_1+l_2)^3}{2l_2^3 c_t (l_0+l_1)^3} \\
F_0 &= F(\mathbf{\Phi}=0) = \frac{-H_{21}(\mathbf{\Phi}=0)}{H_{11}(\mathbf{\Phi}=0)} = \frac{l_1^2(3l_0l_1+3l_0l_2+2l_1l_2+3l_0^2)}{2l_2(l_0+l_1)^3} \\
Q_1 &= G_{31}(\mathbf{\Phi}=0) = -\frac{l_1^2(3l_0+l_1+3l_2)}{(l_0+l_1+l_2)^3} \\
Q_2 &= G_{32}(\mathbf{\Phi}=0) = \frac{l_2^2(3l_0+3l_1+l_2)}{(l_0+l_1+l_2)^3} \\
Z_1 &= G_{21}(\mathbf{\Phi}=0) = -\frac{l_1^2(l_0^2+5l_0l_2-l_1l_0+4l_2^2)}{2(l_0+l_1+l_2)^3} \\
Z_2 &= G_{22}(\mathbf{\Phi}=0) = -\frac{l_2^2(l_0^2+5l_0l_1-l_2l_0+4l_1^2)}{2(l_0+l_1+l_2)^3} \\
P_{11} &= \left.\frac{\partial G_{11}}{\partial \phi_1}\right|_{\mathbf{\Phi}=0} = \frac{l_1^2(l_0+l_2)^2}{(l_0+l_1+l_2)^3} \\
P_{12} &= \left.\frac{\partial G_{11}}{\partial \phi_2}\right|_{\mathbf{\Phi}=0} = -\frac{l_1^2 l_2(l_0+l_2)}{(l_0+l_1+l_2)^3} \\
P_{21} &= \left.\frac{\partial G_{12}}{\partial \phi_1}\right|_{\mathbf{\Phi}=0} = \frac{l_1 l_2^2(l_0+l_1)}{(l_0+l_1+l_2)^3} \\
P_{22} &= \left.\frac{\partial G_{12}}{\partial \phi_2}\right|_{\mathbf{\Phi}=0} = -\frac{l_2^2(l_0+l_1)^2}{(l_0+l_1+l_2)^3} \\
J_1 &= \frac{H_{12}(\mathbf{\Phi}=0)}{\Delta_H(\mathbf{\Phi}=0)} = \frac{l_1^2 l_2^2 c_t(3l_0l_1+3l_0l_2+2l_1l_2+3l_0^2)}{3(l_0+l_1+l_2)^3} \\
J_2 &= \frac{H_{22}(\mathbf{\Phi}=0)}{\Delta_H(\mathbf{\Phi}=0)} = \frac{2l_1^3 c_t(l_0+l_2)^3}{3(l_0+l_1+l_2)^3}.
\end{aligned} \tag{26}$$

Finally, we prove the statement which appears at the end of Section 3 of the submitted paper. It is argued that the optimal links ratio l_1/l_2 which maximizes the net forward motion per period $\overline{X}$ is precisely the inverse of the ratio which minimizes the mechanical work per travel distance $\underline{\lambda}$. The (leading order) expressions for $\overline{X}$ and $\underline{\lambda}$ are given as

$$\overline{X} = \pi\varepsilon^2 F_0 E$$
$$\underline{\lambda} = \frac{J_2\omega_c}{2F_0 E} \tag{27}$$
where $E = \frac{1}{2}(P_{21} - P_{12} + Q_2Z_1 - Q_1Z_2)$ and $\omega_c = kN_0$.

Recall the definition of the transformation Υ, which interchanges between the left and right link lengths l_i and angles ϕ_i. Since the expressions in (27) are defined at $\phi_1 = \phi_2 = 0$, we define the link lengths transformation $\widetilde{\Upsilon}[f(l_1, l_2)] = f(l_2, l_1)$. Our goal is to show that the relation $\underline{\lambda} = c/\widetilde{\Upsilon}[\overline{X}]$ holds for some scalar $c > 0$, which then directly proves the original statement. Note that our goal is to prove this relation by using only the definitions of $\overline{X}$ and $\underline{\lambda}$ in (27) without using the explicit expressions derived under the RFT approximation as in (26).

First, we identify some geometric symmetries underlying the swimmer's equations of motion, which imply that elements of the 2×2 symmetric matrix $\mathbf{H}(\boldsymbol{\Phi})$ in (21) and the 3×2 matrix $\mathbf{G}(\boldsymbol{\Phi})$ in (22) satisfy the relations

$$\begin{aligned}
H_{22}(\boldsymbol{\Phi}) &= \Upsilon[H_{11}(\boldsymbol{\Phi})] \\
H_{21}(\boldsymbol{\Phi}) &= \Upsilon[H_{21}(\boldsymbol{\Phi})] = H_{12}(\boldsymbol{\Phi}) \\
G_{12}(\boldsymbol{\Phi}) &= -\Upsilon[G_{11}(\boldsymbol{\Phi})] \\
G_{22}(\boldsymbol{\Phi}) &= \Upsilon[G_{21}(\boldsymbol{\Phi})] \\
G_{32}(\boldsymbol{\Phi}) &= -\Upsilon[G_{31}(\boldsymbol{\Phi})]
\end{aligned} \tag{28}$$

The relations in (28) are consequences of the geometric symmetries of the swimmer under interchanging between the two links 1 and 2. They also hold in (24) and (23), as a special case, though they are not limited to the RFT formulation. Using the definitions in (26), the following relations are directly implied by (28):

$$\begin{aligned}
Q_2 &= -\widetilde{\Upsilon}[Q_1] \\
Z_2 &= \widetilde{\Upsilon}[Z_1] \\
P_{22} &= -\widetilde{\Upsilon}[P_{11}] \\
P_{21} &= -\widetilde{\Upsilon}[P_{12}] \\
E &= \widetilde{\Upsilon}[E]
\end{aligned} \tag{29}$$

where E is defined in (27). Next, substituting the definitions in (26) into (27) gives

$$\begin{aligned} \overline{X} &= -\pi\varepsilon^2 E \frac{H_{21}(\mathbf{\Phi}=0)}{H_{11}(\mathbf{\Phi}=0)} \\ \underline{\lambda} &= -\frac{kH_{22}(\mathbf{\Phi}=0)}{2EH_{21}(\mathbf{\Phi}=0)} \end{aligned} \tag{30}$$

(Note that $H_{21}(0) < 0$, hence $\overline{X}$ and $\underline{\lambda}$ are both positive). Finally, using the relations (28) and (29) one obtains that $\underline{\lambda} = \dfrac{\pi\varepsilon^2 k}{2\widetilde{\Upsilon}[\overline{X}]}$, which completes the proof.

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