Proof of zero normal force in the RW model

According to equations (35) and (29) in the paper, the normal force f_n along a solution of the RW with initial conditions $\theta(0) = -\frac{\pi}{n}$ and $\dot{\theta}(0) = \dot{\theta}_0$ is

$$f_n(\theta) = \frac{mg}{1+\kappa}\left(\left(3\cos(\alpha+\theta) - \dot{\theta}_0^2 - 2\cos(\alpha - \tfrac{\pi}{n})\right)\cos\theta + \kappa\cos\alpha\right) \quad (1)$$

where $\alpha \in \left(0, \frac{\pi}{2}\right)$, $\kappa \in [0,1]$, $n \in \{4,5,6,...\}$, and θ varies within the interval $I_n = [-\frac{\pi}{n}, \frac{\pi}{n}]$. Normalizing the force units so that $mg = 3(1+\kappa)$, one obtains

$$f_n = \left((\cos(\alpha+\theta) - c)\cos\theta + \frac{\kappa}{3}\cos\alpha\right) = \frac{1}{2}\cos(\alpha+2\theta) - c\cos\theta + b, \quad b, c > 0 \quad (2)$$

where $c = \frac{1}{3}(\dot{\theta}_0^2 + 2\cos(\alpha - \frac{\pi}{n})) > 0$ and $b = \left(\frac{1}{2} + \frac{\kappa}{3}\right)\cos\alpha$. One can see that f_n decreases monotonically upon increasing $\dot{\theta}_0$. We assume that along the periodic solution $\dot{\theta}_0 = \dot{\theta}^*$ the normal force is non-negative, i.e. $f_n(\theta) \geq 0$ for all $\theta \in I_n$. Our goal is to prove that if the minimum of $f_n(\theta)$ within I_n is zero, then it must be attained at the upper endpoint $\theta = \frac{\pi}{n}$. We present two proofs, where each proof holds under different assumptions on the parameters.

1 Proof 1

First, we assume that there exist sufficiently small value of $c = c^* > 0$ for which $f_n(\theta) > 0$ for all $\theta \in I_n$. (Otherwise, no periodic solution of the RW exists that maintains contact). Thus, let $c_{min} > c^*$ denote the minimal value of c for which $f_n(\theta) = 0$ for some $\theta \in I_n$. Therefore, any $c \leq c_{min}$ satisfies

$$c\cos\theta \leq \frac{1}{2}\cos\alpha + \frac{1}{2}\cos(\alpha+2\theta) + b \quad \text{for all } \theta \in I_n. \quad (3)$$

We restrict the parameters α, κ and n by assuming that they satisfy the following inequality, which is crucial for this proof:

$$-9\cos(\alpha + 2\pi/n) + (3+2\kappa)\cos\alpha < 0. \quad (4)$$

We will show later how this assumption affects the generality of the proof. The first and second derivatives of $f_n(\theta)$ are given by:

$$f_n'(\theta) = -\sin(\alpha + 2\theta) + c\sin\theta. \tag{5}$$

$$f_n''(\theta) = -2\cos(\alpha + 2\theta) + c\cos\theta. \tag{6}$$

Using inequality (3) then implies that any $c \le c_{min}$ satisfies

$$f_n''(\theta) \le -\frac{3}{2}\cos(\alpha+2\theta)+b = -\frac{3}{2}\cos(\alpha+2\theta)+\left(\frac{1}{2}+\frac{\kappa}{3}\right)\cos\alpha = w(\theta) \tag{7}$$

for all $\theta \in I_n$. Note that the function $w(\theta)$ in (7) attains its maximum within I_n at the endpoint $\theta=\frac{\pi}{n}$. Using (7), the bound (4) then implies that for $c = c_{min}$ one obtains $f_n''(\theta) \le w(\frac{\pi}{n}) < 0$ for all $\theta \in I_n$. In that case, even if a critical point of $f_n(\theta)$ exists within I_n, it must be a maximum point, hence $f_n(\theta)$ attains its minimum value only at an endpoint of I_n. Finally, substitution of the values of f_n at the endpoints yields

$$f_n(-\frac{\pi}{n}) - f_n(\frac{\pi}{n}) = \frac{1}{2}\cos\left(\alpha - \frac{\pi}{n}\right) - \frac{1}{2}\cos\left(\alpha + \frac{\pi}{n}\right) > 0. \tag{8}$$

This implies that the minimum of $f_n(\theta)$ within I_n is attained at the upper endpoint $\theta=\frac{\pi}{n}$, which completes the proof.

Next, we study the range of parameters for which this proof is valid. For given values of κ and n, the assumption (4) implies an upper bound $\alpha < \alpha_{MAX}$. On the other hand, inequalities (34) and (37) from the paper give upper and lower bounds α_{min} and α_{max} for which the periodic solution is valid. The lower bound $\alpha > \alpha_{min}$ must be satisfied in order to ensure existence of the periodic solution with $\dot{\theta}_0 = \dot{\theta}^*$, while the upper bound $\alpha < \alpha_{min}$ must be satisfied in order to ensure positive contact force $f_n(\theta = \frac{\pi}{n}) > 0$. Figures 1(a,b,c) plot the region $\alpha_{min} < \alpha < \alpha_{max}$ (shaded) and the upper bound $\alpha = \alpha_{MAX}$ (dashed line) as a function of κ for $n = 6, 8, 9$ respectively. It can be seen that for $n \ge 9$, the assumption (4) holds for the entire region $\alpha_{min} < \alpha < \alpha_{max}$. For $n = 6, 7, 8$ the assumption (4)

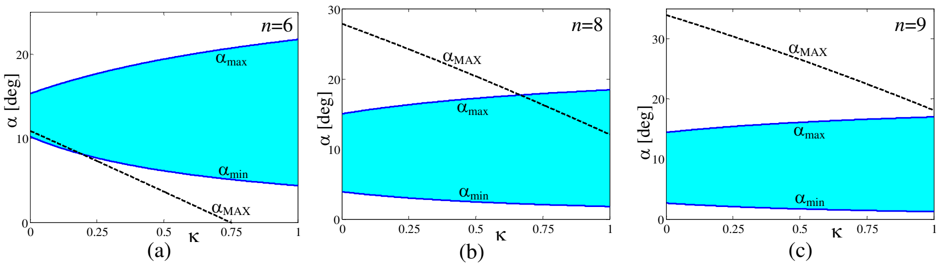

Figure 1: The region $\alpha_{min} < \alpha < \alpha_{max}$ and the bound $\alpha < \alpha_{MAX}$ for (a) $n = 6$, (b) $n = 8$ and (c) $n = 9$.

holds only for part of this region, while for $n = 4, 5$ the assumption (4) is not satisfied at all. Thus, we conclude that this proof holds for $n \geq 9$ and arbitrary κ, α.

2 Proof 2

The second proof takes a slightly different approach, and uses an assumption on the parameters which is less restrictive than (4). First, we define $c_0 = c_0(\alpha, \kappa, n)$ as

$$c_0(\alpha, \kappa, n) = \frac{\cos(\alpha + \frac{2\pi}{n}) + 2b}{2\cos(\frac{\pi}{n})} > 0. \tag{9}$$

This definition implies that for $c{=}c_0$, one obtains $f_n(\theta{=}\frac{\pi}{n}) = 0$. Setting $c{=}c_0$, We now need to prove that the minimum of $f_n(\theta)$ within I_n is attained at the endpoint $(\theta{=}\frac{\pi}{n})$. Instead of using assumption (4), we assume that the parameters α, κ and n satisfy the inequality

$$-2\cos(\alpha - \frac{2\pi}{n}) + \cos(\frac{\pi}{n})c_0(\alpha, \kappa, n) < 0. \tag{10}$$

We will show later that this assumption, which is necessary for the proof, has a minor effect on the generality of the result. We now focus on the interval of nonnegative angles $\theta \in I_+ = [0, \frac{\pi}{n}]$. Define the series of functions

$$g_k(\theta) = -4^k \sin(\alpha + 2\theta) + c_0 \sin\theta \tag{11}$$

for $k = 0, 1, 2, ...$ It can be verified that the series $g_k(\theta)$ satisfies $g_{k+1}(\theta) = -g_k''(\theta)$. Furthermore, $g_k(\theta)$ are related to the odd-order derivatives of the normal force $f_n(\theta)$ as

$$\frac{d^{2k+1}}{d\theta^{2k+1}} f_n(\theta) = (-1)^k g_k(\theta). \tag{12}$$

In particular, for k=0 one obtains $g_0(\theta) = f_n'(\theta)$. It can be proven (details omitted) that $g_0(\theta{=}\frac{\pi}{n}) < 0$. Note that this also implies that $g_k(\theta{=}\frac{\pi}{n}) < 0$ for all k. Additionally, it is easy to see that $g_k(0) < 0$ for all k. Moreover, there exists a sufficiently large k^* which satisfies $g_{k^*}(\theta) < 0$ for all $k > k^*$ and $\theta \in I_+$. This value of k^* is given by

$$k^* > \log_4 \left(\max \left\{ \frac{c_0 \sin\theta}{\sin(\alpha + 2\theta)}, \ \theta \in I_+ \right\} \right). \tag{13}$$

Next, we prove that $g_k(\theta) < 0$ for all $\theta \in I_+$ and for all k, by backward induction in k starting from k=k^*. Suppose that $g_k(\theta)$ for some k is indeed negative for all $\theta \in I_+$. This implies that $g_{k-1}''(\theta) > 0$ for all $\theta \in I_+$. Thus, if $g_{k-1}(\theta)$ has a critical point within $\theta \in I_+$, it must be a minimum point. Therefore, the maximal value of $g_{k-1}(\theta)$ can only be attained at one of the endpoints of I_+. Since $g_{k-1}(0) < 0$ and $g_{k-1}(\frac{\pi}{n}) < 0$, we conclude that $g_{k-1}(\theta) < 0$ for all $\theta \in I_+$. Carrying this induction down to k=0, the relation (12) then implies that $f_n'(\theta) < 0$ for all $\theta \in I_+$, hence $f_n(\theta)$ has no critical point within $\theta \in I_+$.

Next, we focus on the interval of negative angles $\theta \in I_- = [-\frac{\pi}{n}, 0]$. For $k = 0, 1, 2, ...$, define the series of functions

$$h_k(\theta) = -2 \cdot 4^k \cos(\alpha + 2\theta) + c_0 \cos\theta. \tag{14}$$

It can be verified that the series $h_k(\theta)$ satisfies $h_{k+1}(\theta) = -h_k''(\theta)$. Furthermore, $h_k(\theta)$ are related to the even-order derivatives of the normal force

$f_n(\theta)$ as

$$\frac{d^{2k+2}}{d\theta^{2k+2}} f_n(\theta) = (-1)^k h_k(\theta). \tag{15}$$

In particular, for $k = 0$ one obtains $h_0(\theta) = f_n''(\theta)$. It can be proven (details omitted) that $h_0(0) < 0$. Moreover, inequality (10) implies that $h_0(-\frac{\pi}{n}) < 0$. Note that this also implies that $h_k(-\frac{\pi}{n}) < 0$ and $h_k(0) < 0$ for all k, since $h_k(\theta) < h_0(\theta)$ for all $k > 0$ and all $\theta \in I_-$. Moreover, there exists a sufficiently large k^* which satisfies $h_{k^*}(\theta) < 0$ for all $k > k^*$ and $\theta \in I_-$. This value of k^* is given by

$$k^* > \log_4 \left(\max \left\{ \frac{c_0 \cos\theta}{2\cos(\alpha + 2\theta)},\ \theta \in I_- \right\} \right). \tag{16}$$

Next, we prove that $h_k(\theta) < 0$ for all $\theta \in I_-$ and for all k, by backward induction in k starting from k=k^*. Suppose that $h_k(\theta)$ for some k is indeed negative for all $\theta \in I_-$. This implies that $h''_{k-1}(\theta) > 0$ for all $\theta \in I_-$. Thus, if $h_{k-1}(\theta)$ has a critical point within $\theta \in I_-$, it must be a minimum point. Therefore, the maximal value of $h_{k-1}(\theta)$ is attained at the endpoints of I_-. Since $h_{k-1}(0) < 0$ and $h_{k-1}(\frac{\pi}{n}) < 0$, we conclude that $h_{k-1}(\theta) < 0$ for all $\theta \in I_-$. Carrying this induction down to $k = 0$, the relation (15) implies that $f_n''(\theta) < 0$ for all $\theta \in I_-$. Therefore, if $f_n(\theta)$ has a critical point within I_-, it must be a maximum point.

Finally, since $f_n(\theta)$ has no minimum point within I_- and no critical point within I_+, one concludes that its minimum within I_n is attained only at the endpoints $\theta = \pm\frac{\pi}{n}$. Using (8) , it can be shown that the minimum is attained at the upper endpoint θ=$\frac{\pi}{n}$, which completes the proof.

We now study the range of parameters for which this proof is valid. For given values of κ and n, the assumption (10) implies a lower bound $\alpha > \alpha_{MIN}$. On the other hand, inequalities (34) and (37) from the paper impose upper and lower bounds $\alpha_{min} < \alpha < \alpha_{max}$ for which the periodic solution is valid. Figures 2(a,b,c) plot the region $\alpha_{min} < \alpha < \alpha_{max}$ (shaded) and the lower bound $\alpha = \alpha_{MIN}$ (dashed line) as a function of κ for $n = 4, 5, 6$ respectively. It can be seen that for $n \geq 6$, the assumption (10) holds for the entire region $\alpha_{min} < \alpha < \alpha_{max}$. For $n = 4, 5$ the assumption (10) holds almost everywhere, except for a very small part of this region. Furthermore, numerical checks show that even in these small regions where (10) is not satisfied, the statement is still true and $f_n(\theta)$ attains its minimum at the upper endpoint of I_n and has a maximum point within I_- despite the fact that $f_n''(-\frac{\pi}{n}) > 0$, since $f_n'' < 0$ at the critical point where $f_n' = 0$. As an example, Figure 2(d,e,f) plots $f_n(\theta)$, $f_n'(\theta)$ and$f_n''(\theta)$, respectively, for $c = c_0$ and parameter values of $n = 4$, $\kappa = 1$ and $\alpha = 16°$, for which (10) is not satisfied.

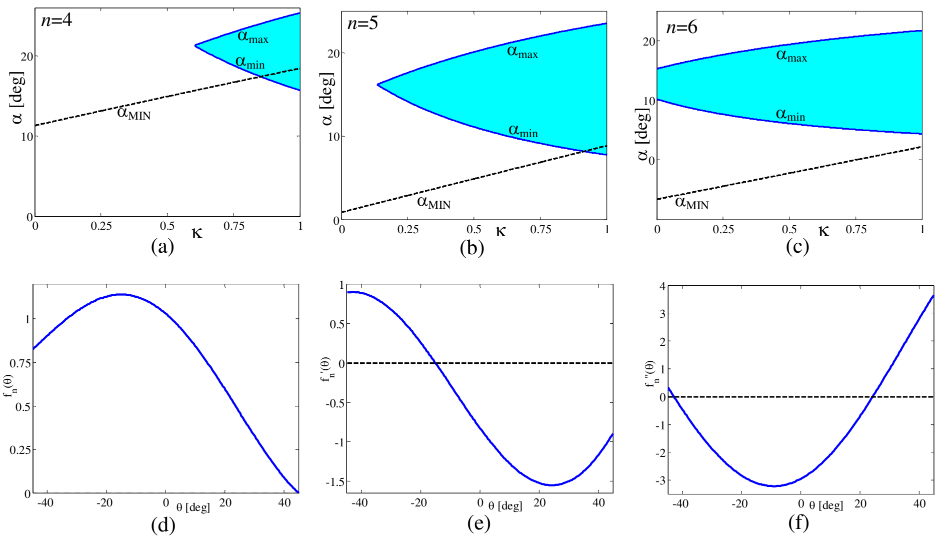

Figure 2: The region $\alpha_{min} < \alpha < \alpha_{max}$ and the bound $\alpha > \alpha_{MIN}$ for (a) $n = 6$, (b) $n = 8$ and (c) $n = 9$; Plots of (d) $f_n(\theta)$, (e) $f_n'(\theta)$ and (f) $f_n''(\theta)$, for $c = c_0$, $n = 4$, $\kappa = 1$ and $\alpha = 16°$.