Supplemental material to: Dynamics and Stability of a Class of Low Reynolds Number Swimmers Near a Wall

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Basic concepts from dynamical systems and control theory

In this section we give a brief overview of basic concepts from dynamical systems and control theory, which are relevant to our study. An *(autonomous) dynamical systems* is given by the vectorial differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{1}$$

where $\mathbf{x} \in I\!\!R^n$ is called the *state vector*, $\mathbf{f}(\mathbf{x})$ is a Lipschitz-continuous vector field, and dot represents derivative with respect to the time t. A *control system* is a set of two equations of the form

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \\ \mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u}) \end{cases} \tag{2}$$

where $\mathbf{u} \in I\!\!R^m$ are the *inputs* and $\mathbf{u} \in I\!\!R^r$ are the *outputs* of the system. Traditionally, in a control problem, one needs to design input functions $\mathbf{u} = \mathbf{u}(t)$, or a *feedback law* $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, which is based on measuring the state variables $\mathbf{x}$ in real-time, in order to obtain a desired motion of the output functions $\mathbf{y}(t)$ under (2), for tasks such as tracking a desired trajectory or stabilization of an equilibrium point. Under a given input $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, substituting into (2) and ignoring the output $\mathbf{y}$, one again obtains a dynamical system of the form (1), where notions such as equilibrium points and their stability can then be studied.

An *equilibrium point* of a dynamical system (1) is a point $\mathbf{x}_e$ such that $\mathbf{f}(\mathbf{x}_e) = 0$. An equilibrium point is $\mathbf{x}_e$ said to be *(locally) stable* if for any arbitrarily small open neighborhood U of $\mathbf{x}_e$, there exists another open neighborhood V, such that for any initial condition $\mathbf{x}(0) \in V$, the solution of (1) satisfies $\mathbf{x}(t) \in U$ for all $t \geq 0$. In addition, $\mathbf{x}_e$ is said to be *(locally) asymptotically stable* if it is locally stable, and for any initial condition $\mathbf{x}(0) \in V$, the solution $\mathbf{x}(t)$ of (1) converges asymptotically to $\mathbf{x}_e$ as $t \to \infty$. If $\mathbf{x}_e$ is not stable it is called *unstable*, and if $\mathbf{x}_e$ is stable but not asymptotically stable, it is sometimes called *marginally stable*. In the vicinity of an equilibrium point $\mathbf{x}_e$, the dynamical system (1) can be approximated by its *linearization* about $\mathbf{x}_e$, which given by

$$\delta\dot{\mathbf{x}} = \mathbf{A}\delta\mathbf{x} \text{ , where } \delta\mathbf{x} = \mathbf{x} - \mathbf{x}_e \text{ and } \mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_e} \tag{3}$$

The asymptotic stability of an equilibrium point $\mathbf{x}_e$ can then be determined by the linearization, as summarized by the following classical theorem.

Theorem 1 (Hartman-Grobman) *Let $\mathbf{x}_e$ be an equilibrium point of a dynamical system (1), and let $\mathbf{A}$ be the linearization matrix defined in (3). Then if all eigenvalues of $\mathbf{A}$ have strictly negative real part, $\mathbf{x}_e$ is locally asymptotically stable, and if all eigenvalues of $\mathbf{A}$ have strictly positive real part, $\mathbf{x}_e$ is unstable.*

Note that in case where $\mathbf{A}$ has eigenvalues with zero real part, the stability of $\mathbf{x}_e$ cannot be fully determined by the linearization. For purely linear dynamical systems of the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, if $\mathbf{A}$ has a single zero eigenvalue, or a single pair of purely imaginary eigenvalues, then it can be shown that the equilibrium point $\mathbf{x}_e = 0$ is marginally stable. This property, however, cannot be directly extended to nonlinear dynamical systems.

Consider now a dynamical system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu), \tag{4}$$

where μ is a parameter of the system that does not change with time. By varying μ, (4) can be regarded as a *one-parameter family of dynamical systems*. Assume that for each constant value of μ, the system (4) has an equilibrium point $\mathbf{x}_e$ that changes continuously with μ. Suppose that one varies the parameter μ continuously, and examines the changes in the eigenvalues of the linearization matrix of (4) at the changing equilibrium point $\mathbf{x}_e(\mu)$. A critical value $\mu = \mu_c$ at which the stability characterization of $\mathbf{x}_e(\mu)$ changes, is called *a bifurcation point*. A bifurcation which is associated with a pair of eigenvalues of $\mathbf{A}$ having zero real part (i.e. purely imaginary) is called a *Hopf Bifurcation*. In that case, the classical theorem by Hopf states that for values of μ in the vicinity of μ_c, there exists a *periodic solution* for (4) around $\mathbf{x}_e$, i.e. a solution $\mathbf{x}(t)$ of (4) that satisfies $\mathbf{x}(t) = \mathbf{x}(t+T)$ at all t, for some $T > 0$. The reader is referred to standard textbooks such as [1] for more details on stability analysis of these periodic solutions, as well as characterization of other types of bifurcations.

The swimmer dynamics as a control system

Recall that the equation of motion for a constant-shape swimmer in viscous fluid is given by the control system

$$\dot{\mathbf{q}} = \mathbf{G}(\mathbf{q})\mathbf{u}, \quad \text{where } \mathbf{G}(\mathbf{q}) = (\mathbf{T}^T\mathbf{R}\mathbf{T})^{-1}\mathbf{T}^T\mathbf{R}\mathbf{E}. \tag{5}$$

In the presence of a plane wall at $y = 0$ [Fig. 1], the equation (5) is invariant under motion along x, and can be reduced to

$$\dot{\mathbf{q}}' = \mathbf{G}'(\mathbf{q}')\mathbf{u} \tag{6}$$

where $\mathbf{q}' = (y, \theta)$ and $\mathbf{G}'$ is the lower 2×2 block of $\mathbf{G}(\mathbf{q})$ in (5).

Swimmer in unbounded fluid - gauge symmetry

The dynamics of the swimmer *in unbounded fluid* has a special geometric structure, called *gauge symmetry*, which implies that the equation of motion is invariant under rigid body transformations. In our setup of swimmers with constant shape, the gauge symmetry greatly simplifies the equation of motion, as follows. Let $\mathbf{U}_{body} = (\mathbf{V}_{body}, \omega_{body})^T$ denote the velocity of the swimmer's body expressed in the body frame $\mathcal{F}_b$. Writing the swimming equation (5) in terms of the body-frame velocity $\mathbf{U}_{body}$ reads $\mathbf{U}_{body} = \mathbf{G}\mathbf{u}$, where $\mathbf{G}$ is now a constant matrix, which is obtained by evaluating $\mathbf{G}(\mathbf{q})$ at a configuration where the two frames $\mathcal{F}_w$ and $\mathcal{F}_b$ are parallel.

Example: the two-sphere swimmer in unbounded fluid

Consider the two-sphere swimmer in unbounded fluid, and denote $\mathbf{U}_{body} = (v_x, v_y, \omega)$ as the body-frame velocity. Using the model in [2] that accounts for far-field hydrodynamic interactions, the equation of motion for the swimmer is given by

$$\begin{pmatrix} v_x \\ v_y \\ \omega \end{pmatrix} = \begin{pmatrix} -k_1 & k_1 \\ 0 & 0 \\ -k_2 & -k_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \text{ where}$$

$$k_1 = \frac{d}{2\hat{d}^3 + 1}, k_2 = \frac{2(8\hat{d}^3) - 9\hat{d}^2 - 4}{6\hat{d}^5 + 32\hat{d}^3 - 51\hat{d}^2 - 16}, \text{ and } \hat{d} = \frac{d}{a}.$$

Note that when taking equal and opposite input velocities $u_1 = -u_2$, the swimmer moves along the x' axis due to its axisymmetry. The *direction of motion* [Fig 3(a)], however, might seem unintuitive, as the same swimmer will move in the reversed direction under inertial effects (see [3] and [4] for discussions of this fact).

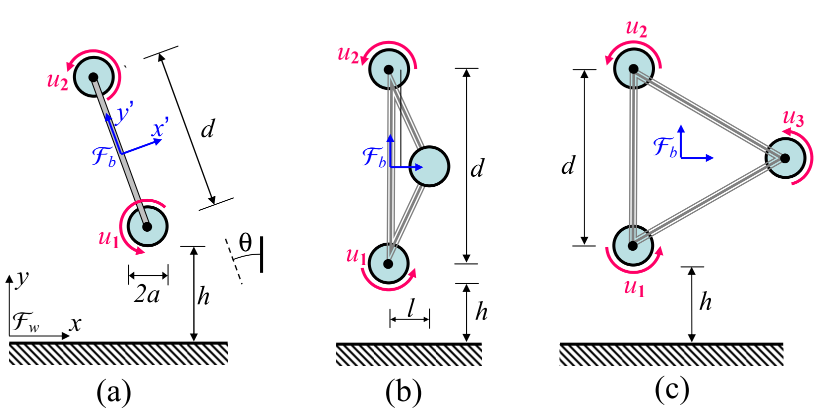

FIG. 1: Drawing of the swimmers. (a) Two-sphere swimmer. (b) 2+1- sphere swimmer. (c) Three-sphere swimmer.

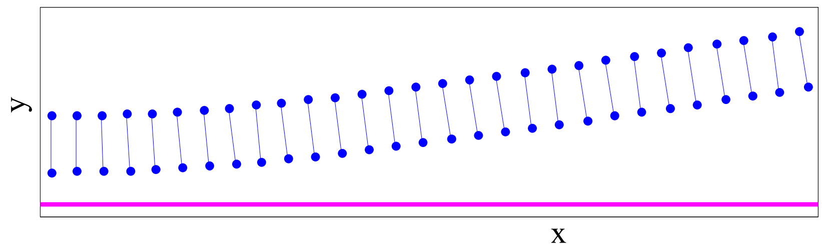

FIG. 2: Snapshots of the motion of the two-sphere swimmer near a wall for $u_1 = -u_2$

The two-sphere swimmer near wall

When the two sphere-swimmer is placed near a wall, taking equal and opposite input velocities $u_1 = -u_2$ no longer results in straight-line motion, and the swimmers is repelled from the wall along an arc [Fig. 2]. The cause of this motion is the loss of axisymmetry due to the presence of the wall, which implies that the sphere closer to the wall now experiences larger hydrodynamic resistance than the other sphere, resulting in body rotation. Nevertheless, in the perpendicular orientation $\theta = 0$, for any given distance from the wall, there exist a specific ratio of input velocities that compensates for the difference in resistances by rotating the sphere closer to the wall more slowly in order to cancel the body rotation. Under these input velocities, the fluid exerts equal and opposite torques on the two spheres, resulting in steady motion parallel to the wall.

Computing the linearization matrix

In order to compute the linearization matrix $\mathbf{A}$ about a given relative equilibrium point $\mathbf{q}'_e = (y_e, \theta_e)$, one has to formulate the derivatives of the matrix $\mathbf{G}'$ with respect to the coordinates y and θ. These derivatives are obtained by applying the chain rule on the definition of $\mathbf{G}(\mathbf{q})$ in (5), and are given by

$$\frac{\partial \mathbf{G}}{\partial y} = \mathbf{M}_b \mathbf{T}^T \mathbf{R} \frac{\partial \mathbf{M}}{\partial y} \mathbf{R}(\mathbf{I} - \mathbf{T}\mathbf{M}_b\mathbf{T}^T\mathbf{R})\mathbf{E}$$

$$\frac{\partial \mathbf{G}}{\partial \theta} = \mathbf{M}_b \left((\mathbf{T}^T \mathbf{R} \frac{\partial \mathbf{M}}{\partial \theta} - \frac{\partial \mathbf{T}}{\partial \theta}^T) \mathbf{R}(\mathbf{I} - \mathbf{T}\mathbf{M}_b\mathbf{T}^T\mathbf{R}) + \mathbf{T}^T\mathbf{R}\frac{\partial \mathbf{T}}{\partial \theta}\mathbf{M}_b\mathbf{T}^T\mathbf{R} \right) \mathbf{E},$$

where $\mathbf{M}_b = (\mathbf{T}^T\mathbf{R}\mathbf{T})^{-1}$ and $\mathbf{I}$ is the identity matrix. The derivatives of the mobility matrix $\mathbf{M} = \mathbf{R}^{-1}$ are computed by direct differentiation of the expressions in [5].

Intuitive explanation for insability of the two-sphere swimmer at $\theta = 0$

As shown in the paper, the characteristic polynomial of the linearization matrix $\mathbf{A}$ at an equilibrium point $\mathbf{q}'_e = (y_e, 0)$ has the form $\Delta_A(\lambda) = \lambda^2 - bc$, which precludes the possibility of asymptotic stability of $\mathbf{q}'_e$. An intuitive explanation of this fact, which is implied by the reversing symmetry, is illustrated in Fig. 3 as follows. Assume that parallel swimming to the left [Fig 3(a)] is asymptotically stable. The reflection symmetry then implies that parallel swimming to the right [Fig 3(b)] is also asymptotically stable. Therefore, under initial perturbation in the orientation angle θ [Fig 3(c)], the motion of the swimmer should converge back to the perpendicular orientation. Exploiting the linearity of the swimming equation (5) in the inputs $\mathbf{u}$, reversing the direction of the inputs will reverse the direction of the body velocity [Fig 3(d)], and the swimmer's orientation then diverges under the same perturbation. Thus, we conclude that parallel swimming to the left [Fig 3(a)] is *unstable*, in contradiction to the initial assumption.

Another interesting observation is that the eigenvalues of $\mathbf{A}$ form a symmetric pair $\pm\lambda$, typical for systems with reversing symmetry [6], where existence of a solution behaving like $e^{\lambda t}$ implies the existence of a reflected and time-reversed solution behaving like $e^{-\lambda t}$.

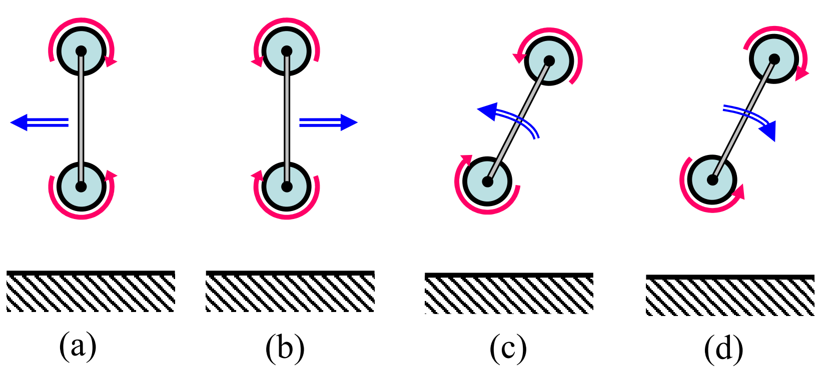

FIG. 3: The two-sphere swimmer near wall - Illustrative explanation of the inherent impossibility of asymptotic stability at $\theta = 0$.

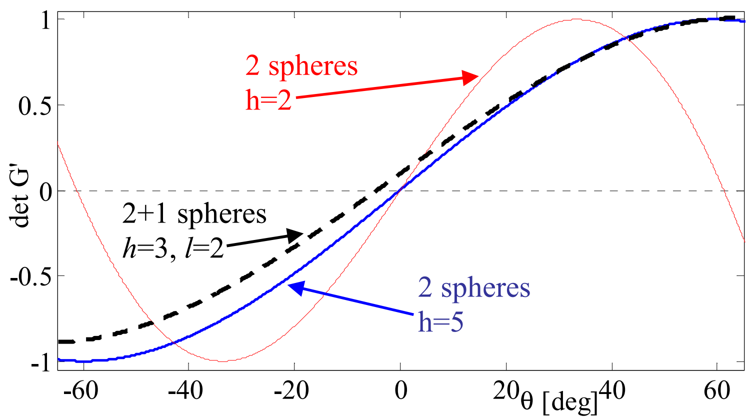

FIG. 4: Plot of $\mathbf{det}\, G'(\theta)$ for the two-sphere swimmer with $h = 5$ (thick blue) and $h = 2$ (thin red), and for the 2+1-sphere swimmer with $h = 3,\ l = 2$ (dashed black).

Parallel motion of the two-sphere swimmer at asymmetric orientations

Since swimming in perpendicular orientation cannot be asymptotically stable, we seek for steady parallel swimming at *asymmetric orientations* of the swimmer, i.e. we try to find equilibrium points of (6) for $\theta \neq 0$. A necessary condition for equilibrium is that the columns of the matrix $\mathbf{G}'(\mathbf{q}')$ are linearly dependent, so that $\mathbf{G}'(\mathbf{q}')\mathbf{u} = 0$ for some input $\mathbf{u} = \mathbf{u}_e$. In the following, we numerically compute $\mathbf{det}\,\mathbf{G}'$ as a function of θ when h is held constant. The results are shown in Fig. 4 for $h = 5$ (thick blue) and $h = 2$ (thin red). For large separations such as $h = 5$ and above, the only zero-crossing is at $\theta = 0$, corresponding to the perpendicular orientation. However, for the smaller separation $h = 2$, there exists an additional zero-crossing point at $\theta = 61.25°$, associated with input vector $\mathbf{u}_e = (1, -1.017)^T$. Numerical computation of the linearization matrix $\mathbf{A}$ and its eigenvalues gives $\lambda_1 = -0.0012$, $\lambda_2 = 0.0008$, hence the equilibrium point $\mathbf{q}'_e$ is *unstable*. Similar results were obtained numerically for other values of d and h, indicating that these additional equilibrium point are always unstable, and correspond to the saddle points appearing in the phase portrait of Fig. 2(b) at the paper.

Parallel motion of the 2+1-sphere swimmer

In the case of the 2+1-sphere swimmer [Fig. 1(b)], the condition for existence of steady parallel swimming is again $\mathbf{det}\,\mathbf{G}' = 0$. As an example, $\mathbf{det}\,\mathbf{G}'$ as a function of θ for $d = 10$, $l = 2$, and $h = 3$ is shown in Fig. 4 (dashed black). Due to the symmetry breaking, the zero crossing point is slightly shifted from $\theta = 0$, giving an equilibrium point $\mathbf{q}'_e = (8.9866, -4.19°)$ under the input $\mathbf{u}_e = (-1, 1.0086)$. The eigenvalues of the associated linearization matrix $\mathbf{A}$ are $\lambda_{1,2} = (-0.3984 \pm 1.5341i) \cdot 10^{-3}$, hence $\mathbf{q}'_e$ is *asymptotically stable*.

Computing region of attraction of stable relative equilibria

We now show that the reversing symmetries in the system, along with existence of periodic motion, enable computation of the region of attraction of the stable relative equilibrium points $\mathbf{q}'_e$, which is the region of all initial conditions in (y, θ) plane that result in convergence to $\mathbf{q}'_e$. In the three-sphere swimmer [Fig 1(c)], let $\mathbf{u}_e(\theta_e)$ be the constant input associated with parallel motion at $\mathbf{q}'_e = (y_e, \theta_e)$ for a given constant y_e. It can then be shown that the one-parameter family of dynamical systems $\dot{\mathbf{q}}' = \mathbf{G}'(\mathbf{q})\mathbf{u}_e(\theta_e)$ possesses a reversing symmetry with respect to reflecting both θ and the parameter θ_e about 30°. For example, trajectories of the system with $\theta_e = 0°$ are obtained from trajectories of the system with $\theta_e = 60°$ [Fig. 4(b) in the paper] by reversing time and reflecting θ about 30°. Therefore, the system with $\theta_e = 0°$, that has a stable equilibrium point $\mathbf{q}'_e = (0, 0°)$ also has an *unstable periodic orbit*, which is precisely the closed curve enclosing the *region of attraction* of the stable equilibrium $\mathbf{q}'_e$ in (y, θ) plane. Note that a similar result also holds for the 2+1-sphere swimmer [Fig 1(b)], which possesses a reversing symmetry with respect to reversing the signs of θ and the parameter l.

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