Supplementary material to: Dynamics and stability of Purcell's three-link microswimmer near a wall

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In this supplementary document, the dynamic equations of motion of the chain-of-spheres three-link swimmer are formulated. A sketch of the swimmer appears in Fig. 1. The swimmer is neutrally buoyant, and is submerged in a quiescent viscous fluid. Since the Reynolds number is assumed to be negligibly small due to the small scale of the swimmer, the motion of the fluid is governed by Stokes equation [1]. The no-slip boundary conditions imply that the fluid velocity matches the velocity of the swimmer on its solid boundary surfaces. The fluid exerts drag forces on the swimmer through the viscous stresses acting at the boundary surfaces. Since the motion of the swimmer and the fluid is assumed to be quasi-steady (no inertial effects), the net forces and torques acting on the swimmer are zero at all times. The conditions described above compose a very complicated system of differential equations describing the fluid-solid interaction. Nevertheless, a fundamental property of these equations is the existence of a linear relation between forces acting on the submerged solid bodies and their velocities, called the *resistance relation* [1]. This relation enables explicit formulation of the swimmer's equation of motion, as described next.

Consider the swimmer model in Fig. 1, which is comprised of a collection of ten identical spheres of equal radius that are connected by thin rigid rods. For simplicity, it is assumed that the motion of the swimmer is confined to the (x, y)-plane. Let $\mathbf{r}_i \in I\!R^2$ denote the planar position of the ith sphere center. Let $\mathbf{v}_i = \dot{\mathbf{r}}_i$ denote the translational velocity of the ith sphere, and let ω_i denote its angular velocity about the z-axis. Let $\mathbf{q} = (\mathbf{r}, \theta)^T$ where $\mathbf{r} = (x, y)^T$ denote the planar position and orientation of a reference frame attached to the swimmer's central link, and let $\mathbf{s} = (\phi_1, \phi_2)^T$ denote the angles of the two distal links with respect to the central link (see Fig. 1). For a given configuration of the swimmer $(\mathbf{q}, \mathbf{s})$, the positions of the spheres $\mathbf{r}_i$ are determined by rigid-body kinematic relations. Moreover, the velocities $\mathbf{v}_i$ and ω_i are given explicitly as

$$\mathbf{v}_i = \begin{cases} \dot{\mathbf{r}} + \dot{\theta}\mathbf{J}(\mathbf{r}_i - \mathbf{r}) - \dot{\phi}_1\mathbf{J}(\mathbf{r}_i - \mathbf{r}_4) & i = 1,2,3 \\ \dot{\mathbf{r}} + \dot{\theta}\mathbf{J}(\mathbf{r}_i - \mathbf{r}) & i = 4,5,6,7 \\ \dot{\mathbf{r}} + \dot{\theta}\mathbf{J}(\mathbf{r}_i - \mathbf{r}) + \dot{\phi}_2\mathbf{J}(\mathbf{r}_i - \mathbf{r}_7) & i = 8,9,10 \end{cases} \tag{1}$$

where $\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Augmenting all the linear and angular velocities in a column vector $\mathbf{U} = (\mathbf{v}_1, \dots \mathbf{v}_{10}, \omega_1, \dots \omega_{10})$, the relation (1) can be rewritten in matrix form as

$$\mathbf{U} = \mathbf{T}\dot{\mathbf{q}} + \mathbf{E}\dot{\mathbf{s}}, \tag{2}$$

where $\mathbf{T}$ is a 30×3 matrix and $\mathbf{E}$ is a 30×2 matrix, both depending on ϕ_1, ϕ_2 and θ. The net force and torque exerted on the ith sphere by the viscous fluid are denoted $\mathbf{f}_i \in I\!R^2$ and $\tau_i \in I\!R$, respectively. The total force and torque acting on the swimmer's body, denoted $\mathbf{f}_b$ and τ_b, can be expressed with respect to the reference frame attached to

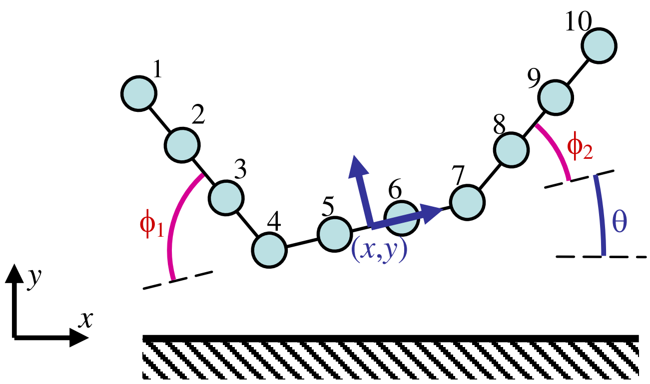

FIG. 1: The chain-of-spheres three-link swimmer model near a wall.

the swimmer's central link as [3]

$$\mathbf{f}_b = \mathbf{f}_1 + \ldots + \mathbf{f}_{10} \text{ and } \tau_b = \sum_{i=1}^{10} \tau_i + (\mathbf{r}_i - \mathbf{r})^T \mathbf{J}^T \mathbf{f}_i. \tag{3}$$

Augmenting the forces and torques acting on the individual spheres and on the swimmer's body in column vectors $\mathbf{F} = (\mathbf{f}_1, \ldots \mathbf{f}_{10}, \tau_1, \ldots \tau_{10})$ and $\mathbf{F}_b = (\mathbf{f}_b, \tau_b)$, it can be verified that (3) can be written in matrix form as

$$\mathbf{f}_b = \mathbf{T}^T \mathbf{F}, \tag{4}$$

where $\mathbf{T}$ is the same matrix as in (2). The linear relation between the velocities of the spheres and the forces and torques exerted on them by the fluid can be formulated as

$$\mathbf{F} = \mathbf{R}\mathbf{U}. \tag{5}$$

$\mathbf{R}$ is a 30×30 matrix called the resistance matrix [1], which depends on the positions of the spheres and encapsulates all the hydrodynamic interactions between the spheres, the fluid and the boundary. In general, the resistance matrix for multiple interacting rigid bodies cannot be formulated exactly, but it can be approximated under certain scaling assumptions. In this work, we use the results of Swan and Brady [2], who formulated an approximation of the *mobility matrix*, defined as $\mathbf{M} = \mathbf{R}^{-1}$, for multiple spheres in a fluid domain which is bounded by an infinite plane wall. Using the relations (2), (4) and (5), the requirement that the net forces and torque on acting on the swimmer's body must vanish at all times gives

$$\mathbf{F}_b = \mathbf{T}^T \mathbf{F} = \mathbf{T}^T \mathbf{R}\mathbf{U} = \mathbf{T}^T \mathbf{R}(\mathbf{T}\dot{\mathbf{q}} + \mathbf{E}\dot{\mathbf{s}}) = 0. \tag{6}$$

Inverting the relationship (6) then gives the swimmer's equation of motion

$$\dot{\mathbf{q}} = \mathbf{G}(\mathbf{q}, \mathbf{s})\dot{\mathbf{s}}, \text{ where } \mathbf{G}(\mathbf{q}, \mathbf{s}) = -(\mathbf{T}^T \mathbf{R}\mathbf{T})^{-1}\mathbf{T}^T \mathbf{R}\mathbf{E}. \tag{7}$$

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[1] J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics* (Prentice-Hall, New Jersey, 1965).
[2] J. W. Swan and J. F. Brady, Phys. Fluids **19**, 113306 (2007).
[3] It is assumed that the viscous forces exerted by the fluid on the thin connecting rods are negligible compared to the forces acting on the spheres