Zeno Stability of the Set-Valued Bouncing Ball

Yizhar Or and Andrew R. Teel

Abstract

Hybrid dynamical systems are systems consisting of both continuous-time and discrete-time dynamics. A fundamental phenomenon that is unique to hybrid systems is *Zeno behavior*, where the solution involves an infinite number of discrete transitions occurring in finite time, as best illustrated in the classical example of a bouncing ball. In this note, we study the hybrid system of the **set-valued bouncing ball**, for which the continuous-time dynamics is set-valued. This system is typically used for deriving bounds on the solution of nonlinear single-valued hybrid systems in a small neighborhood of a Zeno equilibrium point in order to establish its local stability. We utilize methods of Lyapunov analysis and optimal control to derive a necessary and sufficient condition for Zeno stability of the set-valued bouncing ball system and to obtain a tight bound on the Zeno time as a function of initial conditions.

I. Introduction

Hybrid dynamical systems are systems that consist of both continuous-time and discrete-time dynamics [8], [23]. A fundamental phenomenon that is unique to hybrid systems is *Zeno behavior*, where the solution involves an infinite number of discrete transitions occurring in finite time. The classical example of Zeno behavior is the *bouncing ball* system, describing the one-dimensional motion of a ball bouncing on a flat ground, where the collisions of the ball with the ground are modeled as rigid-body impacts. It is easily shown that if the impacts are not perfectly elastic, the ball displays Zeno behavior for any given initial conditions. Moreover, derivation of a closed-form expression for the finite accumulation time (Zeno time) as a function of initial conditions is straightforward.

Y. Or is with the Faculty of Mechanical Engineering, Technion - Israel Institute of Technology, Haifa 32000, Israel; `izi@tx.technion.ac.il`. A. R. Teel is with the ECE Department, University of California Santa Barbara, CA 93106-9560; `teel@ece.ucsb.edu`. His research is supported in part by AFOSR under grant A9550-09-1-0203 and NSF under grants ECS-0622253, CNS-0720842, and ECCS-0925637

Zeno behavior has recently gained increasing interest, in works studying conditions for existence of Zeno behavior [6], [12], [24] and its relation to asymptotic stability [1], [9], [15], [20]. In particular, some works have focused on *Lagrangian hybrid systems*, which model unilaterally constrained mechanical systems undergoing impacts [3]. In this class of models, the configuration q of the mechanical system is restricted to satisfy a unilateral constraint function $h(q) \geq 0$, representing rigid-body contact. Zeno solutions in such systems converge to limit points $(q, \dot{q})$ satisfying $h(q) = 0,\ Dh(q)\dot{q} = 0$, called *Zeno equilibria*. In general, the nonlinearity of the system precludes the derivation of explicit expression for the Zeno limit point and Zeno time of a solution under given initial conditions. Moreover, even determining whether the solution under a given initial condition is Zeno is not obvious. It was recently shown in [13] that a necessary and sufficient condition for existence of Zeno solutions in the vicinity of a Zeno equilibrium point $x^* = (q^*, \dot{q}^*)$ is that $\ddot{h}(x^*) < 0$, where $\ddot{h}(x^*)$ is the second-order time derivative of the constraint function along trajectories of the system's continuous-time dynamics, evaluated at x^*. Moreover, the same condition also implies *local stability* of x^* [15]. The physical interpretation of this condition is that the hybrid dynamics of the constraint function $h(q(t))$ is *locally similar to that of a bouncing ball*. However, this stability criterion only guarantees *existence* of a small neighborhood of initial conditions near x^* that lead to Zeno solutions. Two fundamental questions that naturally arise are: Can one obtain an *explicit* expression for a neighborhood of initial conditions all leading to Zeno solutions? Can one derive bounds on the Zeno times and Zeno limit points of solutions starting at a given neighborhood? Answering these questions may prove useful for several applications, such as establishing bounds on numerical errors in simulation of Zeno solutions [2], [17], and deriving conditions for existence of periodic solutions with Zeno behavior in Lagrangian hybrid systems [16]. A key simplifying step towards addressing these two questions is to focus solely on the dynamics of the constraint function $h(q)$ along trajectories of the system. In a given neighborhood U of a Zeno equilibrium point, the nonlinear dynamics of $h(q(t))$ can be replaced with the *set-valued* dynamics given by the second-order differential inclusion $\ddot{h} \in [-a_{max}, -a_{min}]$, where a_{max} and a_{min} are obtained by computing *bounds* on the exact dynamics of h within the neighborhood U. When h vanishes, a discrete jump occurs according to the impact law $\dot{h} \to -e\dot{h}$, where e is the Newtonian coefficient of restitution. These two components constitute the hybrid system studied in this note — the *set-valued bouncing ball* (**SVBB**). Interestingly, while in the classical single-valued bouncing ball, $e < 1$ implies that all solutions are Zeno, this is not true in the set-valued case.

The stability of unilaterally constrained mechanical system has also been studied in the non-smooth mechanics literature, e.g. [4], [14]. In particular, a paper which is closely related to our work is that of Heimsch and Leine [11]. This work studies the problem of a ball bouncing on a vibrating table and derives a condition for finite-time attractivity of its solutions. Despite the differences between this problem and the $\mathbf{SVBB}$, the proof in [11] only uses *bounds* on the ball-table relative acceleration. Thus, the condition derived in [11] also applies to the Zeno stability discussed here for the $\mathbf{SVBB}$ system. The proof in [11] uses a non-standard Lyapunov argument, under which the Lyapunov function is allowed to increase in continuous time but its values at the discrete impact times form a decreasing series. Additionally, [11] derives an upper bound on the finite attraction time — the Zeno time.

The contributions of this note are as follows. First, we formulate the set-valued hybrid dynamics of the $\mathbf{SVBB}$ system, which generalizes the specific system analyzed in [11], and accounts for all possible scenarios of nonlinearities or uncertainties in single-valued and possibly non-autonomous hybrid systems. Second, we propose a Lyapunov function which is strictly decreasing along the continuous-time dynamics and use it in order to derive a condition under which the set-valued bouncing ball is Zeno and asymptotically stable, which agrees with the condition given in [11]. The stability condition is also proven here by using optimal control arguments, which provide an intuitive explanation of the "most unstable" solution. Finally, the exact tight bound on the Zeno time of all possible solutions of the $\mathbf{SVBB}$ is explicitly derived by using optimal control analysis. Results on optimal control for hybrid systems have appeared in the literature, e.g. [7], [22]. However, we show here that the generality of these works is not needed because of the specific structure of the problem. Finally, it is important to note that our $\mathbf{SVBB}$ model is incomplete in the sense that it only captures the hybrid dynamics under intermittent contact and impacts. After the solution reaches a Zeno equilibrium point, the physical motion enters a different phase of persistent contact, whose dynamics strongly depends on contact forces and on the assumed model of friction. In order to model the dynamics in this phase, which is not captured by the $\mathbf{SVBB}$ model, one has to consider the *completed hybrid system* model [2], [17], or alternatively, use more sophisticated complementarity formulations that describe non-smooth force-velocity relations (e.g. [3], [14]).

II. PRELIMINARIES

In this section we give our basic terminology of set-valued hybrid systems, define the notion of uniform Zeno stability, and formulate the problem of the set-valued bouncing ball.

A. Hybrid systems

Let $F, G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be set-valued mappings and $C, D \subset \mathbb{R}^n$ be sets. We consider hybrid systems of the form

$$\mathcal{H} : \quad \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{cases} \tag{1}$$

For more background on hybrid systems in this framework, see [8].

A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if $E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \; ... \leq t_J$. It is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ...J\})$ is a compact hybrid time domain. Equivalently, E is a hybrid time domain if E is a union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, with the "last" interval possibly of the form $[t_j, T)$ with T finite or $T = +\infty$. A *hybrid arc* is a function ϕ whose domain, denoted dom ϕ, is a hybrid time domain and such that for each $j \in \mathbb{N}$, $t \to \phi(t, j)$ is locally absolutely continuous on $I_j := \{t \,|\, (t, j) \in \text{dom } \phi\}$. A hybrid arc ϕ is *complete* if its domain, dom ϕ, is unbounded. A hybrid arc ϕ is a *solution to the hybrid system* $\mathcal{H}$ if $\phi(0, 0) \in C \cup D$ and

(i) for all $j \in \mathbb{N}$ such that $t \in \text{int}(I_j)$: $\phi(t, j) \in C$ and $\dot{\phi}(t, j) \in F(\phi(t, j))$;

(ii) for all $(t, j) \in \text{dom } \phi$ such that $(t, j+1) \in \text{dom } \phi$: $\phi(t, j) \in D$ and $\phi(t, j+1) \in G(\phi(t, j))$.

A solution ϕ is *maximal* if there does not exist a solution ψ with dom $\phi \subset$ dom ψ, dom $\phi \neq$ dom ψ, $\phi(t, j) = \psi(t, j)$ for all $(t, j) \in \text{dom } \phi$. Complete solutions are maximal.

B. Zeno solutions and uniform Zeno stability

A hybrid arc ϕ is called *Zeno* if it is complete but $T(\phi) := \sup\{t \in \mathbb{R}_{\geq 0} \,|\, \exists j \text{ s.t. } (t, j) \in \text{dom } \phi\}$ is finite. In short, ϕ is Zeno if it experiences infinitely many jumps in finite (ordinary) time. For a given initial condition $x_0 \in C \cup D$, let $Z_{max}(x_0)$ denote the supermum of $T(\phi)$ over all possible solutions ϕ satisfying $\phi(0, 0) = x_0$. Finally, denoting $|\cdot|_{\mathcal{A}}$ as the distance from a set $\mathcal{A} \subset \mathbb{R}^n$ in the Euclidean norm, we define the uniform Zeno stability of $\mathcal{A}$ as follows.

Definition 1 (Uniform Zeno stability [9]): A compact set $\mathcal{A} \subset \mathbb{R}^n$ is called *uniformly Zeno asymptotically stable* ($\mathbf{UZAS}$) for the hybrid system $\mathcal{H}$ if the following hold:

(a) each solution ϕ to (1) is bounded and Zeno, and also satisfies $|\phi(t,j)|_{\mathcal{A}} \to 0$ as $t+j \to \infty$, $(t,j) \in \text{dom } \phi$.

(b) for each $\varepsilon_a, \varepsilon_b > 0$ there exists $\delta > 0$ such that, for each maximal solution ϕ to (1) with $|\phi(0,0)|_{\mathcal{A}} \leq \delta$ one has $|\phi(t,j)|_{\mathcal{A}} \leq \varepsilon_a$ for all $(t,j) \in \text{dom } \phi$ and $T(\phi) < \varepsilon_b$.

C. The set-valued bouncing ball

The set-valued bouncing ball ($\mathbf{SVBB}$) is a hybrid system with state $x \in \mathbb{R}^2$ and data

$$\begin{aligned} &C = \left\{x \in \mathbb{R}^2 : x_1 \geq 0\right\}, \qquad D = \left\{x \in \mathbb{R}^2 : x_1 = 0,\ x_2 \leq 0\right\}, \\ &F(x) = \left\{ \begin{bmatrix} x_2 \\ -a \end{bmatrix}, a \in [a_{min}, a_{max}] \right\}, \qquad G(x) = \begin{bmatrix} 0 \\ -ex_2 \end{bmatrix}, \end{aligned} \tag{2}$$

where $e \in (0,1)$ and $0 < a_{min} \leq a_{max}$.

In the case where $a_{min} = a_{max} = g$, the system (2) simply describes the single-valued hybrid dynamics of the classical bouncing ball, where x_1 is the height of the ball above the ground, x_2 is the ball's vertical velocity, and g is the acceleration of gravity. The jump rule $x_2 \to -ex_2$ in (2) represents a rigid-body impact with the ground which induces an instantaneous jump in the velocity, where e is called *Newton's coefficient of restitution*. In that case, it is well known that the origin $\boldsymbol{O} = (0,0)$ is uniformly Zeno asymptotically stable, and obtaining a closed-form expression for the Zeno time for given initial condition is straightforward. However, in the general case where $a_{min} \neq a_{max}$, these issues are more complicated. Thus, the main focus of this note is on the following two problems:

1) Given the hybrid system of the $\mathbf{SVBB}$ in (2), find conditions on a_{min}, a_{max} and e guaranteeing that the origin $\boldsymbol{O}$ is $\mathbf{UZAS}$.
2) Under these conditions, find an explicit expression for the maximal Zeno time $Z_{max}(x_0)$ for any given initial condition $x_0 \in C \cup D$.

III. Lyapunov characterization of Zeno Stability

In this section we address the first problem presented above from the viewpoint of Lyapunov analysis for hybrid systems as described in [9], [20].

A. Lyapunov characterization of the $\mathbf{SVBB}$

We now review a result from [9], [20] on a Lyapunov characterization of uniform Zeno stability in hybrid systems, applied to our model of the $\mathbf{SVBB}$ system. The analysis in [9], [20] is fairly general and characterizes several types of stability of sets in hybrid systems of the form (1). The following proposition presents simplified version of a result from [9], [20], which focuses on the $\mathbf{SVBB}$ model and gives a condition for uniform Zeno stability of the origin $\boldsymbol{O} = (0, 0)$.

Proposition 1 ([9]): Consider the hybrid system of the set-valued bouncing ball given in (2). Then the origin $\boldsymbol{O}$ is $\mathbf{UZAS}$ if there exist a constant $c > 0$, a set $U \supset C \cup D \setminus \{\boldsymbol{O}\}$ and a *Lyapunov function* $V : U \to \mathbb{R}_{\geq 0}$ that is continuously differentiable on U, and radially unbounded and positive definite with respect to $\boldsymbol{O}$ on $U \cap (C \cup D)$, such that

$$\begin{array}{lll} \text{(i)} & \text{For all } x \in C \setminus \{\boldsymbol{O}\} \text{ and } f \in F(x), & \langle \nabla V(x), f \rangle \leq -c \\ \text{(ii)} & \text{For all } x \in D \setminus \{\boldsymbol{O}\}, & V(G(x)) \leq V(x). \end{array}$$

The idea of the proof is based on the fact that for a given initial condition $\phi(0, 0) = x_0$, (i) implies that the time-derivative of $V(x)$ along continuous parts of the solutions of the hybrid system satisfies $\dot{V} \leq -c$, while (ii) implies that $V(x)$ is not increasing on discrete jumps. Therefore, $V(x)$ satisfies $V(\phi(t, j)) \leq V(x_0) - ct$ for all $(t, j) \in \text{dom } \phi$. Since $V(x)$ is positive definite and vanishes only at $x = \boldsymbol{O}$, any solution[1] will reach $\boldsymbol{O}$ in an ordinary time which is bounded by $V(x_0)/c$.

B. A condition for Uniform Zeno stability of the $\mathbf{SVBB}$

We now use Proposition 1 to derive a necessary and sufficient condition for uniform Zeno stability of the set-valued bouncing ball. It was already shown in [9] that in the classical bouncing ball example, i.e. with $a_{min} = a_{max} = g$, $\boldsymbol{O}$ possesses uniform Zeno stability for any coefficient

[1] Note that in general, Lyapunov conditions like those in Proposition 1 do not guarantee local existence of solutions and thus, not all maximal solutions are necessarily complete. Completeness of solutions requires additional conditions [9], which are satisfied automatically in the particular case of the $\mathbf{SVBB}$.

of restitution $e < 1$. This was proven by choosing a Lyapunov function of the form $V(x) = x_2 + k\sqrt{x_2^2/2 + gx_1}$. Intuitively, this Lyapunov function is a rescaled combination of the ball's total mechanical energy with an additional term proportional to the velocity x_2. In the set-valued case, the condition for **UZAS** depends on e, as well as on the bounds a_{min} and a_{max}, as summarized in the following theorem.

Theorem 1: *The origin of the set-valued bouncing ball whose dynamics is described by (2) possesses uniform Zeno stability if and only if the following condition holds:*

$$e^2\alpha < 1, \text{ where } \alpha = \frac{a_{max}}{a_{min}}. \tag{3}$$

Note that the condition (3) agrees with the condition derived in [11], where a_{min} and a_{max} are taken as the lower and upper bounds on the time-dependent acceleration of the ball relative to the vibrating table.

Proof: First, we prove the "if" part. Consider the function $V : U \to \mathbb{R}_{\geq 0}$ defined by

$$V(x) = \kappa x_2 + \sqrt{W(x)}, \text{ where } W(x) = \frac{1}{2p(x_2)}x_2^2 + x_1, \text{ and } U = \left\{x \in I\!R^2 : W(x) > 0\right\}, \tag{4}$$

$$\text{where} \quad p(x_2) = \begin{cases} a_{max} & \text{if } x_2 \leq 0 \\ a_{min} & \text{if } x_2 > 0 \end{cases} \quad \text{and} \quad \kappa = \left(\frac{1}{\sqrt{2a_{max}}} - \frac{e}{\sqrt{2a_{min}}}\right)\frac{1}{(1+e)},$$

Note that even though $p(x_2)$ is piecewise-defined, $V(x)$ is continuous on dom V and continuously differentiable on dom $V \setminus \{\boldsymbol{O}\}$. Under condition (3), we have $\kappa > 0$, hence it can be verified that $V(x)$ is positive definite with respect to $\boldsymbol{O}$ on $U \cap (C \cup D)$ (see [18]). Moreover, the choice of κ implies that for all $x \in D$, one has $V(G(x)) = V(x)$. That is, $V(x)$ does not change at discrete jumps. Finally, it can be verified that for all $x \in C \setminus \{0\}$ and $f \in F(x)$, one gets $\langle \nabla V(x), f \rangle \leq -\kappa a_{min}$ (see [18]). That is, $V(x)$ is strictly decreasing along the continuous-time flow. Therefore, $V(x)$ is a Lyapunov function satisfying the conditions of Proposition 1, and $\boldsymbol{O}$ is **UZAS**.

In order to prove the "and only if" part, consider the candidate Lyapunov function $W(x)$ given in (4), which is positive definite on $C \cup D$. Assume that condition (3) is violated. It can then be shown that for all $x \in D \setminus \{\boldsymbol{O}\}$, we have $W(G(x)) - W(x) \geq 0$. In the continuous part, choosing a solution $f \in F(x)$ for which $a = p(x_2)$, gives $\langle \nabla W(x), f \rangle = 0$. Therefore, one can construct a solution with arbitrarily small initial condition, for which $W(x)$ is non-decreasing. Thus, this solution cannot converge to the origin. ■

IV. Optimal control analysis of Zeno stability

In this section, we utilize techniques of optimal control theory in order to analyze the system of the set-valued bouncing ball. First, we reproduce the proof of Theorem 1 in a way that gives physical interpretation of the condition for uniform Zeno stability. Then we derive an exact expression for the maximum Zeno time $Z_{max}(x_0)$. We begin by reviewing some basic terminology and concepts of optimal control theory, and, in particular, of Pontryagin's maximum principle.

A. Review of Pontryagin's maximum principle

We now give a brief summary of Pontryagin's maximum principle. The presentation here is based on standard textbooks on optimal control theory such as [5]. Consider a control system

$$\dot{x} = f(x, u), \tag{5}$$

where $x = (x_1, \ldots x_n) \in I\!R^n$, $u \in \Omega \subseteq I\!R^m$, and Ω is a convex set of admissible controls. A solution to (5) on a time interval $[\tau_0, \tau_f]$ is a pair $(x(t), u(t))$ satisfying (5) and $u(t) \in \Omega$ for all $t \in [\tau_0, \tau_f]$. The initial and final conditions of $x(t)$ are denoted $x_0 = x(\tau_0)$ and $x_f = x(\tau_f)$. For a given control system (5), the *optimal control problem* is defined as finding a solution to (5) under given initial conditions x_0, which maximizes a given cost function depending on the end condition[2] $P(x_f, \tau_f)$. This solution, denoted $(x^*(t), u^*(t))$, is called the *optimal solution* associated with the given cost function. Note that the end condition x_f, as well as the end time τ_f, may be either specified or left as free parameters of optimization.

The optimal control problem can be formulated as a problem in calculus of variations, and its solution is based on the classical notion of Pontryagin's maximum principle, which is stated as follows. First, define the *co-state vector* $\lambda(t) \in I\!R^n$. Next, define the system's *Hamiltonian*, given by $H(x, u, \lambda, t) = \lambda(t)^T f(x, u)$. The co-state dynamic equation is then given by

$$\dot{\lambda} = -\frac{\partial H}{\partial x}. \tag{6}$$

[2]Many textbooks also consider an integral cost function of the form $J = \int_{\tau_0}^{\tau_f} g(x, u, t)dt$. This cost function can be easily incorporated into the formulation here by augmenting the state vector x with an additional variable z whose dynamics is given by $\dot{z} = g(x, u, t)$. The cost function is then simply given by $P = z(\tau_f)$.

The optimal control input $u^*(t)$ is given by

$$u^*(t) = \text{argmax}_u\, H(x(t), u(t), \lambda(t), t) \tag{7}$$

for $t \in [\tau_0, \tau_f]$. The optimal solution is obtained by solving the coupled equations (5), (6), and (7) under boundary conditions $x(\tau_0) = x_0$ and $x(\tau_f) = x_f$. If the end condition for $x_{i,f} = x_i(\tau_f)$ is not specified for some $i \in \{1 \dots n\}$, an alternative end condition for λ_i is given by

$$\lambda_i(\tau_f) = \frac{\partial P}{\partial x_{i,f}}. \tag{8}$$

B. *Formulating the* $\mathbf{SVBB}$ *as an optimal control problem*

We now formulate the continuous part of the dynamics of the $\mathbf{SVBB}$ as an optimal control problem, as follows. The control system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \quad \text{where } u \in [-a_{max}, -a_{min}], \tag{9}$$

represents the set-valued differential equation $\dot{x} \in F(x)$, where $F(x)$ is given in (2). We view the initial and final times τ_0 and τ_f as the endpoints of a time interval $[t_j, t_{j+1}]$ in a solution $\phi(t, j)$ of the $\mathbf{SVBB}$ system. The initial condition of (9) is thus given by $x(\tau_0) = (0, \nu)$ for some $\nu \geq 0$. One end condition is specified, namely $x_1(\tau_f) = 0$. However, the end time τ_f, as well as $x_2(\tau_f)$, which corresponds to the ball's terminal velocity, are both unspecified. The Hamiltonian of this system is given by $H(x, \lambda, u, t) = \lambda_1 x_2 + \lambda_2 u$. Using (6), the co-state dynamic equations are then given by $\dot{\lambda}_1 = 0,\ \dot{\lambda}_2 = -\lambda_1$, indicating that $\lambda_1(t)$ is constant and $\lambda_2(t)$ is a linear function of t. Pontryagin's maximum principle (7) then implies that for $\lambda_2(t) \neq 0$, the optimal control input $u^*(t)$ can be either $-a_{min}$ or $-a_{max}$, depending only on the sign of $\lambda_2(t)$. Since $\lambda_2(t)$ is a linear function of t, it has at most one zero-crossing point in the time interval $[\tau_0, \tau_f]$. Therefore, the optimal control $u^*(t)$ is piecewise-constant, with *at most one switching time*. The proofs of all results in the rest of this section will build on this setup for deriving optimal solutions under different choices of a cost function.

C. *Proof of Zeno stability condition via optimal control analysis*

We now reproduce the proof of Theorem 1 which gives the condition for $\mathbf{UZAS}$ of the $\mathbf{SVBB}$, by using optimal control analysis.

Proof [of Theorem 1]: The key idea of the proof is to consider the sequence of velocities x_2 at the discrete times t_j in all possible solutions $\phi(t,j)$, find the "most unstable" possible sequence, and require that it decays asymptotically to zero as $j \to \infty$. For a given solution $\phi(t,j) = (x_1(t,j), x_2(t,j))$ of the **SVBB** system under initial condition $\phi(0,0) = x_0$, denote $v_j = x_2(t_j, j)$ for $j \in \{1, 2 \ldots\}$. Physically, v_j is the post-impact velocity of the ball right after the jth collision with the ground. By construction, one has $v_j \geq 0$. First, we show that for any possible solution $\phi(t,j)$, the sequence of v_j satisfies

$$v_{j+1} < e\sqrt{\alpha}v_j, \tag{10}$$

where α is defined in (3). In order to prove (10), we consider the optimal control system (9), which describes the evolution of $\phi(t,j)$ in a specific time interval $[t_j, t_{j+1}]$. The initial condition is given by $x(\tau_0) = (0, \nu)$ where $\nu > 0$, and the end condition is $x_1(\tau_f) = 0$, where the end velocity $x_2(\tau_f) = x_{2f}$ is unspecified. We seek for the solution $(x^*(t), u^*(t))$ that maximizes the absolute value of x_{2f}. Since $x_{2f} < 0$, the cost function to be maximized is chosen as $P(x_f, \tau_f) = -x_{2f}$. The end condition on λ_2 in (8) gives $\lambda_2(\tau_f) = -1$, and the maximum principle (7) then implies that $u^*(\tau_f) = -a_{max}$. Since $u^*(t)$ can switch between a_{max} and a_{min} at no more than a single point, one can write $u^*(t) = -a_{min}$ for $t \in [\tau_0, \tau_s)$ and $u^*(t) = -a_{min}$ for $t \in [\tau_s, \tau_f]$, where τ_s is an unknown switching time. By substitution of $u = u^*(t)$ and direct integration of (9) under the given initial and end conditions, one can solve for the end time τ_f by substituting $x_1(\tau_f) = 0$, and then obtain the terminal velocity x_{2f}. The resulting expressions are

$$\begin{aligned} x_{2f} &= -\sqrt{(\nu - a_{min}\tau_s)^2 + 2a_{max}(\nu\tau_s - a_{min}\tau_s^2/2)} \\ \tau_f &= \tau_s + \frac{\nu - a_{min}\tau_s + \sqrt{(\nu - a_{min}\tau_s)^2 + 2a_{max}(\nu\tau_s - a_{min}\tau_s^2/2)}}{a_{max}}. \end{aligned} \tag{11}$$

Solving $dx_{2f}/d\tau_s = 0$ it can then be shown that x_{2f} attains a minimum of $x_2^*(\tau_f) = -\sqrt{\frac{a_{max}}{a_{min}}}\nu$, for the critical switching time $\tau_s^* = \nu/a_{max}$. One can also verify that $x_2^*(\tau_s^*) = 0$. The physical interpretation of this optimal solution is selecting the "slowest" acceleration $\dot{x}_2 = -a_{min}$ when the ball is on its way up, i.e. $x_2 > 0$, and the "fastest" acceleration $\dot{x}_2(t) = -a_{max}$ for the way down, i.e. $x_2 > 0$. Under these selections, one attains maximal hitting velocity at the impact. Interestingly, the optimal input is precisely $u^* = -p_2(x_2)$, where $p_2(x_2)$, defined in (4), was used in the derivation of the Lyapunov function. Finally, setting $\nu = v_j$, applying the jump rule $v_{j+1} = -ex_2(t_{j+1}, j) = -ex_{2f}$ and using the definition for α in (3) completes the proof of the bound in (10). As a result, for any solution $\phi(t,j)$, the sequence v_j is bounded by a geometric series as

$$v_j \leq v_1 \left(e\sqrt{\alpha}\right)^{j-1}. \tag{12}$$

In order to get asymptotic convergence of $\phi(t,j)$ to $\boldsymbol{O}$, the sequence v_j must decay to zero. That is, the factor of the upper-bounding geometric series in (12) must be less than one, which is precisely condition (3) in Theorem 1. Note that this condition is *necessary and sufficient*, as we proved that the bound in (10) is tight, since the particular solution $\phi(t,j)$ with $a = p(x_2)$ satisfies the bound in (12) as an equality. Finally, it is shown below that all possible solutions $\phi(t,j)$ have a finite upper bound on ordinary time $T(\phi) < \infty$, which depends linearly on v_1. Therefore, under condition (3), all possible solutions are Zeno and converge to $\boldsymbol{O}$ in finite time, which, by varying the initial condition, can be made arbitrarily small. ■

D. Tight bound on Zeno time of the SVBB

We now present the second main contribution of this note, which is an exact tight bound on the Zeno time of solutions of the **SVBB** system under any given initial condition. As a first step, we focus on initial conditions of the form $\phi(0,0) = (0,\nu)$ for $\nu > 0$, corresponding to the ball starting initially on the ground with nonzero upward velocity. It is shown in [11] that an upper bound on the Zeno time under initial condition of this form is given by

$$Z_{max}\left((\nu,0)\right) \leq \frac{2\nu}{a_{min}(1-\sqrt{\alpha})}. \tag{13}$$

The explanation for this bound is as follows. Given a post-impact velocity v_j at an impact time t_j, the maximal flight duration until the next impact is obtained by choosing the "slowest acceleration" as $t_{j+1} - t_j \leq 2v_j/a_{min}$. On the other hand, using (12), v_j is bounded by $v_j \leq \nu(e\sqrt{\alpha})^j$. Combining these two bounds and using the formula for the sum of a geometric series yields the upper bound in (13). However, this explanation makes it clear that the bound in (13) *can never be a tight bound*. The reason is that the time duration between impacts is maximized by taking constant acceleration $a = a_{min}$, while the upper bound on v_j is obtained by taking a non-constant acceleration (more precisely, $a = p(x_2)$). Therefore, a solution that exactly attains the upper bound in (13) necessarily does not exist. (Another non-tight upper bound for $Z_{max}(x_0)$ can be derived from the Lyapunov function in (4), see [18].) In the following, we derive the *exact tight bound* on the Zeno time, that is, we give an explicit expression for $Z_{max}(x_0)$, as summarized in the following lemma.

Lemma 1: *Consider all possible solutions $\phi(t,j)$ of the* $\mathbf{SVBB}$ *system under initial condition $\phi(0,0) = x_0 = (0,\nu)$, where $\nu > 0$. Assuming that condition (3) is satisfied, all solutions are Zeno, and their maximal Zeno time $Z_{max}(x_0)$ is given by*

$$Z_{max}(x_0) = 2\frac{1+e}{1-e^2\alpha} \cdot \frac{\nu}{a_{min}}. \tag{14}$$

Moreover, there exists a particular solution $\phi^*(t,j)$ such that $T(\phi^*) = Z_{max}(x_0)$.

Proof: Recall that under condition (3), Theorem 1 implies that all possible solutions $\phi(t,j)$ are Zeno. We therefore seek for an *optimal* solution ϕ that maximizes $T(\phi)$ for given ν. Let $\phi^*_\nu(t,j)$ denote this solution. We now make two key observations, as follows. The first observation is that any "tail" of an optimal solution is also an optimal solution. Therefore, denoting $\phi^*_\nu(t_k,k) = (0,v_k)$ for some $k \in I\!N$, one obtains $\phi^*_{v_k}(t,j) = \phi^*_\nu(t+t_k, j+k)$. The second observation is that the $\mathbf{SVBB}$ system satisfies the property of *homogeneity* (cf. [10], see also [21] which uses slightly different terminology). In particular, it can be verified that for any $c > 0$, $\phi(t,j)$ is a solution of (2) if and only if $M(c)\cdot\phi(t/c,j)$ is also a solution of (2), where $M(c) = \mathsf{diag}(c^2,c)$. Therefore, for any $c > 0$ one gets $\phi^*_{c\nu}(t,j) = M(c)\cdot\phi^*_\nu(t/c,j)$. In words, if T^* is the maximal Zeno time under initial velocity ν which is attained by the solution $\phi^*(t,j)$, then scaling the initial velocity ν by c results in a maximal Zeno time T^*/c which is attained by the scaled solution $M(c)\cdot\phi^*(t/c,j)$. Combining these two observations together implies the existence of a scalar $\eta \in (0,1)$ such that for any $(t,j) \in \text{dom } \phi^*_\nu$, one has

$$\phi^*_\nu(t,j) = M(\eta^j)\cdot\phi^*_\nu(\eta^{-j}(t-t_j),0). \tag{15}$$

That is, the behavior of the optimal solution in the jth interval of ordinary time is identical to its behavior in the first time interval up to scaling of magnitude and time. Therefore, the problem of finding the optimal solution $\phi^*_\nu(t,j)$ reduces to solving an optimal control problem on the first time interval $[0,t_1]$ only. The scalar η is then given by $\eta = v_1/\nu = -ex^*_2(t_1)/\nu$, and the remainder of the optimal solution is simply obtained by using (15). In particular, since the discrete times t_j in the optimal solution $\phi^*_\nu(t,j)$ satisfy the geometric series relation $t_{j+1} - t_j = \eta^j t_1$, the Zeno time is given by the sum $T(\phi^*) = \frac{t_1}{1-\eta} = \frac{t_1}{1+ex^*_2(t_1)/\nu}$. Consider again the control system (9), which represents the solution $\phi^*_\nu(t,j)$ in the time interval $[0,t_1]$. We now solve an optimal control problem, where the cost function to be maximized is the Zeno time $T(\phi^*)$, formulated here as $P(x_f, \tau_f) = \frac{\tau_f}{1+ex_{2f}/\nu}$. Using (8), the end condition for λ_2 is given by $\lambda_2(\tau_f) = -\frac{e\tau_f\nu}{(\nu+ex_{2f})^2} < 0$. Therefore, the maximum principle (7) implies that $u^*(\tau_f) = -a_{max}$. Thus, the optimal input satisfies $u^*(t) = -a_{min}$ for $t \in [\tau_0, \tau_s)$ and $u^*(t) = -a_{min}$ for $t \in [\tau_s, \tau_f]$, where τ_s is an unknown switching time. Substituting $u = u^*(t)$, integrating (9), and using the condition $x_1(\tau_f) = 0$, the expressions for the terminal time τ_f and velocity x_{2f} are given in (11). Expressing the cost function $P(x_f, \tau_f)$ in terms of τ_s and solving $dP/d\tau_s = 0$, it can then be shown that the maximal cost is attained for $\tau_s^* = 2\frac{\nu}{a_{min}} \cdot \frac{1+e}{1+2e+e^2\alpha}$. Substitution of $\tau_s = \tau_s^*$ into the expression for P then gives the maximal Zeno time in (14). ■

Finally, we now use Lemma 1 to establish the exact tight bound on the Zeno time of the $\mathbf{SVBB}$ system under *any* initial condition, as summarized in the following theorem.

Theorem 2: *Consider all possible solutions $\phi(t, j)$ of the $\mathbf{SVBB}$ system under initial condition $\phi(0, 0) = x_0 = (h_0, v_0)$. Assuming that condition (3) is satisfied, all solutions are Zeno, and their maximal Zeno time $Z_{max}(x_0)$ is given by*

$$Z_{max}(x_0) = \begin{cases} \dfrac{v_0 + \sigma U_{0min}}{a_{min}} & v_0 \geq v_c \\ \dfrac{v_0 + U_{0max}(1 + \beta\alpha)}{a_{max}} & v_0 < v_c \end{cases} \tag{16}$$

$$\text{where} \quad U_{0max} = \sqrt{v_0^2 + 2a_{max}h_0}\ ,\ U_{0min} = \sqrt{v_0^2 + 2a_{min}h_0}\ ,\ v_c = -\sqrt{\frac{2a_{min}h_0}{\sigma^2 - 1}}$$

$$\alpha = \frac{a_{max}}{a_{min}}\ ,\ \beta = 2e\frac{1+e}{1-e^2\alpha}\ ,\ \sigma = \sqrt{1 + 2\beta + \alpha\beta^2}. \tag{17}$$

Moreover, there exists a particular solution $\phi^*(t, j)$ such that $T(\phi^*) = Z_{max}(x_0)$.

Proof: We are seeking for the solution $\phi^*(t, j)$ that maximizes the Zeno time $T(\phi)$ under initial condition $x_0 = (h_0, v_0)$. Consider the "tail" of the solution $\phi^*(t, j)$ for $t \geq t_1$, which has initial condition $(0, v_1)$. Using Lemma 1, the maximal Zeno time of this solution, denoted T_1^*, is obtained by substituting $\nu = v_1$ in (14). Therefore, one only needs to consider the first time interval $[0, t_1]$ of $\phi^*(t, j)$ as an optimal control problem of the system (9) where $\tau_f = t_1$, and maximize the cost function $t_1 + T_1^*$. The tail's initial velocity v_1 is related to the end condition of the control system (9) via the relation $v_1 = -ex_{2f}$. Therefore, using the expression for T_1^* and the definition of β in (17), the cost function is $P(x_f, \tau_f) = \tau_f - \beta\frac{x_{2f}}{a_{min}}$. Since (8) gives

$\lambda_2(t_f) = -\frac{\beta}{a_{min}} < 0$, the maximum principle (7) implies that $u^*(\tau_f) = -a_{max}$. Thus, the optimal input is taken as $u^*(t) = -a_{min}$ for $t \in [\tau_0, \tau_s)$ and $u^*(t) = -a_{min}$ for $t \in [\tau_s, \tau_f]$, where τ_s is an unknown switching time. Substituting $u = u^*(t)$, integrating (9), and using the condition $x_1(\tau_f) = 0$, the expressions for the terminal time τ_f and velocity x_{2f} are given in (11). Expressing the cost function $P(x_f, t_f)$ in terms of τ_s only and solving $dP/d\tau_s = 0$, it can be shown that maximal cost is attained for the switching time given by $\tau_s^* = \dfrac{\sigma v_0 + U_{0min}}{\sigma a_{min}}$, where U_{0min} and σ are defined in (17). In order to get a nonnegative switching time $\tau_s^* \geq 0$, v_0 must satisfy $\sigma v_0 + U_{0min} \geq 0$. In case where $\sigma v_0 + U_{0min} < 0$, the optimal solution is obtained by taking $\tau_s^* = 0$, so that $u^*(t) = -a_{max}$ at all times, without switching. Finally, direct substitution of the optimal solution into the cost function gives the piecewise-defined expression in (16), where the condition $\sigma v_0 + U_{0min} \geq 0$ is reformulated as $v_0 \geq v_c$. ■

V. Conclusion

In this note we studied the hybrid dynamics of the set-valued bouncing ball. We used Lyapunov analysis and optimal control techniques to obtain a necessary and sufficient condition for Zeno stability, and derived an exact bound on the maximal Zeno time as for given initial conditions. The results are useful for obtaining bounds on Zeno solutions of Lagrangian hybrid systems, as demonstrated in our recent work [19]. Two future directions for extension of the results are as follows. First, consideration of set-valued dynamics that are more complicated than the linear dynamics of the $\mathbf{SVBB}$ may enable a closer approximation of true nonlinear hybrid systems, in order to obtain tighter bounds on their Zeno solutions. Second, utilization of the results for planning cyclic tasks that involve Zeno behavior in robotic systems, in the spirit of [16], as well as stabilization and control of such tasks, are challenging open problems.

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