The Set-Valued Bouncing Ball

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TECHNICAL REPORT

Abstract

Hybrid dynamical systems are systems consisting of both continuous-time and discrete-time dynamics. A fundamental phenomenon that is unique to hybrid systems is *Zeno behavior*, where the solution involves an infinite number of discrete transitions occurring in finite time, as best illustrated in the classical example of a bouncing ball. In this note, we study the hybrid system of the **set-valued bouncing ball**, for which the continuous-time dynamics is set-valued. Such systems are typically used for deriving bounds on the solution of nonlinear single-valued hybrid systems in a small neighborhood of a Zeno equilibrium point in order to establish its local stability. We utilize methods of Lyapunov analysis and optimal control to derive a necessary and sufficient condition for Zeno stability of the set-valued bouncing ball system and to obtain a tight bound on the Zeno time as a function of initial conditions.

I. Introduction

Hybrid dynamical systems are systems that consist of both continuous-time and discrete-time dynamics [11], [15], [20], [28]. They are used to model a wide range of dynamical systems such as biological and chemical processes, coordination of multiple air vehicles, computer-integrated control systems, and robotic systems that involve intermittent contacts. A fundamental phenomenon that is unique to hybrid systems is *Zeno behavior*, where the solution involves an infinite number of discrete transitions occurring in finite time. The classical example of Zeno behavior is the *bouncing ball* system, describing the one-dimensional motion of a rigid ball bouncing on a flat ground, where the collisions of the ball with the ground are modeled as rigid-body impacts with a Newtonian coefficient of restitution. In that case, one can easily show that as long as the impacts are not perfectly elastic, the system displays Zeno behavior for any given initial condition, converging to a limit point where the ball lies at rest on the ground. Moreover, derivation of a closed-form expression for the finite accumulation time (Zeno time) as a function of initial conditions is straightforward.

Zeno behavior has recently gained increasing interest, in works studying conditions for existence of Zeno behavior [1], [8], [16], [26], [30], [31] and its relation to asymptotic stability [2], [13], [21], [24]. In particular, some works have focused on *Lagrangian hybrid systems*, which model unilaterally constrained mechanical systems undergoing

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impacts [6]. In this class of models, the configuration q of the mechanical system is restricted to satisfy a unilateral constraint function $h(q) \geq 0$, representing rigid-body contact. Zeno solutions in such systems typically converge to limit points $(q, \dot{q})$ satisfying $h(q) = 0,\ Dh(q)\dot{q} = 0$, called *Zeno equilibria*. A key fact is that generally, the nonlinearity of the system precludes the derivation of explicit expression for the Zeno limit point and Zeno time of a solution under a given initial condition. Moreover, even determining whether the solution under a given initial condition is Zeno is not obvious. It was recently shown in [18] that a necessary and sufficient condition for existence of Zeno solutions in the vicinity of a Zeno equilibrium point $x^* = (q^*, \dot{q}^*)$ is that $\ddot{h}(x^*) < 0$, where $\ddot{h}(x^*)$ is the second-order time derivative of the constraint function along trajectories of the system's continuous-time dynamics, evaluated at x^*. Moreover, the same condition also implies *local stability* of x^* [21]. The physical interpretation of the condition $\ddot{h}(x^*) < 0$ is that the hybrid dynamics of $h(q(t))$ is *locally similar to that of a bouncing ball*. A key limitation of this stability criterion is that it only guarantees *existence* of a small neighborhood of initial conditions near x^* that lead to Zeno solutions. Two fundamental questions that naturally arise are: Can one obtain an *explicit* expression for a neighborhood of initial conditions all leading to Zeno solutions? Can one derive bounds on the Zeno times and Zeno limit points of solutions starting at a given neighborhood? Answering these questions may prove useful for several applications. For example, many practical simulations of Lagrangian hybrid systems carry executions beyond Zeno points by truncating the solution after a finite number of discrete transitions and projecting it onto the constraint surface [3], [23]. In that case, the explicit bounds described above can be utilized to derive bounds on the numerical error incurred by the truncation. Another example is in the control of bipedal robots by inducing a periodic orbit on the robot's hybrid system model [17], [29], where the impacts are traditionally assumed to be *perfectly plastic*. Using the more realistic model of non-plastic impacts essentially leads to Zeno solutions, and the explicit bounds described above can then be utilized for deriving conditions for existence of a periodic orbit with Zeno behavior in such systems, without requiring explicit computation of the Zeno limit point [22].

A key simplifying step towards addressing the two questions delineated above is to focus only on the dynamics of the constraint function $h(q)$ along trajectories of the system. In a given neighborhood U of a Zeno equilibrium point, the nonlinear *single-valued* dynamics of $h(q(t))$ can be replaced by the *set-valued* dynamics given by the second-order differential inclusion $\ddot{h} \in [-a_{max}, -a_{min}]$, where a_{max} and a_{min} are obtained by computing bounds on the exact dynamics of h within the neighborhood U. When h vanishes, a discrete jump occurs according to the impact law $\dot{h} \to -e\dot{h}$, where e is the Newtonian coefficient of restitution. These two components constitute the hybrid system we study in this work — the *set-valued bouncing ball* ($\mathbf{SVBB}$). Interestingly, while in the classical single-valued bouncing ball, $e < 1$ implies that all solutions are Zeno, this is not true in the set-valued case.

The contributions of this work are as follows. First, we derive a condition under which the set-valued bouncing ball is Zeno and asymptotically stable. We study the system using two different approaches — Lyapunov analysis and optimal control theory — and prove the Zeno stability criterion with both methods. Second, we derive an exact upper bound on the Zeno time of all possible solutions under a given initial condition. While Lyapunov analysis only provides a conservative bound, we show that by using optimal control techniques one obtains the exact tight bound. Results on optimal control for hybrid systems have appeared in the literature, including [27] and [9]. However, the generality of that work is not needed here because of the specific structure of the problem. In particular, our work is inspired by [21], [22], which derived conservative bounds on Zeno time and Zeno limit point in Lagrangian hybrid systems via optimal control techniques, and by the work in [13], [24] which studied stability characterizations in set-valued hybrid systems, and derived Lyapunov conditions for Zeno stability. Finally, we wish to point out that stability characterization of differential inclusions via optimal control analysis was also studied in [19] in the context of switched systems.

II. Preliminaries

In this section we give our basic terminology of set-valued hybrid systems, define the notion of uniform Zeno stability, and formulate the problem of the set-valued bouncing ball.

A. Hybrid systems

Let $F, G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be set-valued mappings and $C, D \subset \mathbb{R}^n$ be sets. We consider hybrid systems of the form

$$\mathcal{H}: \quad \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{cases} \tag{1}$$

For more background on hybrid systems in this framework, see [5], [10], and [12].

A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if $E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ldots \leq t_J$. It is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0,T] \times \{0, 1, \ldots J\})$ is a compact hybrid time domain. Equivalently, E is a hybrid time domain if E is a union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, with the "last" interval possibly of the form $[t_j, T)$ with T finite or $T = +\infty$. A *hybrid arc* is a function ϕ whose domain dom ϕ is a hybrid time domain and such that for each $j \in \mathbb{N}$, $t \to \phi(t, j)$ is locally absolutely continuous on $I_j := \{t \mid (t, j) \in \text{dom } \phi\}$. A hybrid arc ϕ is *complete* if its domain, dom ϕ, is unbounded. A hybrid arc ϕ is a *solution to the hybrid system* $\mathcal{H}$ if $\phi(0, 0) \in C \cup D$ and

(S1) for all $j \in \mathbb{N}$ such that I_j has nonempty interior and for almost all $t \in I_j$,

$$\phi(t, j) \in C, \quad \dot{\phi}(t, j) \in F(\phi(t, j));$$

(S2) for all $(t, j) \in \text{dom } \phi$ such that $(t, j+1) \in \text{dom } \phi$,

$$\phi(t, j) \in D, \quad \phi(t, j+1) \in G(\phi(t, j)).$$

A solution ϕ is *maximal* if there does not exist a solution ψ with dom $\phi \subset$ dom ψ, dom $\phi \neq$ dom ψ, $\phi(t, j) = \psi(t, j)$ for all $(t, j) \in \text{dom } \phi$. Complete solutions are maximal.

B. Zeno solutions and uniform Zeno stability

A hybrid arc ϕ is called *Zeno* if it is complete but

$$T(\phi) := \sup\{t \in \mathbb{R}_{\geq 0} \mid \exists j \text{ s.t. } (t, j) \in \text{dom } \phi\}$$

is finite. In short, ϕ is Zeno if it experiences infinitely many jumps in finite (ordinary) time. In such terminology, the "tail" of the hybrid arc, or even the whole arc itself, may consist of infinitely many instantaneous jumps. For a given initial condition $x_0 \in C \cup D$, let $Z_{max}(x_0)$ denote the supermum of $T(\phi)$ over all possible solutions ϕ satisfying $\phi(0,0) = x_0$. Finally, denoting $|\cdot|_{\mathcal{A}}$ as the distance from a set $\mathcal{A} \subset \mathbb{R}^n$ in the Euclidean norm, we define the uniform Zeno stability of $\mathcal{A}$ as follows.

Definition 1 (Uniform Zeno stability [13]): A compact set $\mathcal{A} \subset \mathbb{R}^n$ is called *uniformly Zeno asymptotically stable* (**UZAS**) for the hybrid system $\mathcal{H}$ if the following hold:

(a) each solution ϕ to (1) is bounded and Zeno, and also satisfies $|\phi(t,j)|_{\mathcal{A}} \to 0$ as $t + j \to \infty$, $(t,j) \in \operatorname{dom} \phi$.

(b) for each $\varepsilon_a, \varepsilon_b > 0$ there exists $\delta > 0$ such that, for each maximal solution ϕ to (1) with $|\phi(0,0)|_{\mathcal{A}} \le \delta$ one has $|\phi(t,j)|_{\mathcal{A}} \le \varepsilon_a$ for all $(t,j) \in \operatorname{dom} \phi$ and $T(\phi) < \varepsilon_b$.

C. The set-valued bouncing ball

The set-valued bouncing ball (**SVBB**) is a hybrid system with state $x \in \mathbb{R}^2$ and data

$$
\begin{aligned}
C &= \left\{x \in \mathbb{R}^2 : x_1 \ge 0\right\}, \\
F(x) &= \left\{ \begin{bmatrix} x_2 \\ -a \end{bmatrix}, a \in [a_{min}, a_{max}] \right\}, \\
D &= \left\{x \in \mathbb{R}^2 : x_1 = 0,\ x_2 \le 0\right\}, \\
G(x) &= \begin{bmatrix} 0 \\ -e x_2 \end{bmatrix},
\end{aligned} \tag{2}
$$

where $e \in (0,1)$ and $0 < a_{min} \le a_{max}$.

In the case where $a_{min} = a_{max} = g$, the system (2) simply describes the single-valued hybrid dynamics of the classical bouncing ball, where x_1 is the height of the ball above the ground, x_2 is the ball's vertical velocity, and g is the acceleration of gravity. The jump rule $x_2 \to -e x_2$ in (2) represents a rigid-body impact with the ground which induces an instantaneous jump in the velocity, where e is called *Newton's coefficient of restitution*, and $1 - e^2$ is the fraction of kinetic energy dissipated through the collision. In that case, it is well known that the origin $\boldsymbol{O} = (0,0)$ is uniformly Zeno asymptotically stable, and obtaining a closed-form expression for the Zeno time for given initial condition is straightforward. However, in the general case where $a_{min} \ne a_{max}$, these issues are more complicated. Thus, the main focus of this work is on the following two problems:

1) Given the hybrid system of the **SVBB** in (2), find conditions on a_{min}, a_{max} and e guaranteeing that the origin $\boldsymbol{O}$ is **UZAS**.
2) Under these conditions, find an explicit formulation for the maximal Zeno time $Z(x_0)$ for any given initial condition $x_0 \in C \cup D$.

III. LYAPUNOV CHARACTERIZATION OF ZENO STABILITY

In this section we address the two problems presented above from the viewpoint of Lyapunov analysis for hybrid systems as described in [13], [24].

A. Lyapunov characterization of the $\mathbf{SVBB}$

We now review a result from [13], [24] on a Lyapunov characterization of uniform Zeno stability in hybrid systems, focusing on our example of the $\mathbf{SVBB}$ system. The analysis in [24] is fairly general and discusses Lyapunov characterization of several types of stability of a compact set $\mathcal{A}$ for a general hybrid system of the form (1). In our particular setup given in (2), we focus on uniform Zeno stability of the origin $\boldsymbol{O} = (0, 0)$ in $\mathbb{R}^2$. Therefore, we present here a basic and simplified version of a result from [13], which is summarized in the following proposition.

Proposition 1 ([13]): Consider the hybrid system of the set-valued bouncing ball given in (2). Then the origin $\boldsymbol{O}$ is $\mathbf{UZAS}$ if there exist a constant $c > 0$ and a *Lyapunov function* $V : C \cup D \to \mathbb{R}_{\geq 0}$ that is continuously differentiable and radially unbounded and positive definite with respect to $\boldsymbol{O}$, such that

$$\begin{array}{lll} \text{(i)} & \text{For all } x \in C \setminus \{\boldsymbol{O}\} \text{ and } f \in F(x), & \langle \nabla V(x), f \rangle \leq -c \\ \text{(ii)} & \text{For all } x \in D \setminus \{\boldsymbol{O}\}, & V(G(x)) \leq V(x). \end{array}$$

The idea of the proof is based on the fact that for a given initial condition $\phi(0, 0) = x_0$, (i) implies that the time-derivative of $V(x)$ along continuous parts of the solutions of the hybrid system satisfies $\dot{V} \leq -c$, while (ii) implies that $V(x)$ is not increasing on discrete jumps. Therefore, $V(x)$ satisfies $V(\phi(t, j)) \leq V(x_0) - ct$ for all $(t, j) \in \text{dom } \phi$. Since $V(x)$ is positive definite and vanishes only at $x = \boldsymbol{O}$, any solution will reach $\boldsymbol{O}$ in an ordinary time which is bounded by $V(x_0)/c$. Note that Lyapunov conditions like those in Proposition 1 do not guarantee local existence of solutions and thus, in general, it is not possible to conclude that maximal solutions are complete. Conditions guaranteeing local existence of solutions are given in [13]. These conditions are automatically satisfied in the particular case of the $\mathbf{SVBB}$.

B. A condition for Uniform Zeno stability of the $\mathbf{SVBB}$

We now use Proposition 1 to derive a necessary and sufficient condition for uniform Zeno stability of the set-valued bouncing ball. It was already shown in [13] that in the classical single-valued bouncing ball example, i.e. with $a_{min} = a_{max} = g$, $\boldsymbol{O}$ possesses uniform Zeno stability. This was proven by choosing a Lyapunov function of the form $V(x) = x_2 + k\sqrt{x_2^2/2 + gx_1}$. Intuitively, this Lyapunov function is a rescaled combination of the ball's total mechanical energy with an additional term proportional to the velocity x_2, guaranteeing the uniform decrease of $V(x)$ along flows. Note that in this case, uniform Zeno stability of $\boldsymbol{O}$ was guaranteed for any coefficient of restitution $e < 1$. However, in the set-valued case this is no longer true, and the condition for $\mathbf{UZAS}$ depends on e, as well as on the bounds a_{min} and a_{max}, as summarized by the following theorem.

Theorem 1: *The origin of the set-valued bouncing ball whose dynamics is described by (2) possesses uniform Zeno stability if and only if the following condition holds:*

$$e^2\alpha < 1, \text{ where } \alpha = \frac{a_{max}}{a_{min}}. \tag{3}$$

Proof: The proof is based on choosing a Lyapunov function according to Proposition 1. First, we prove the "if" part, as follows. Consider the candidate Lyapunov function $V : \text{dom } V \to \mathbb{R}$ defined by

$$V(x) = \kappa x_2 + \sqrt{W(x)}, \text{ where } W(x) = \frac{1}{2p(x_2)}x_2^2 + x_1, \tag{4}$$

where

$$p(x_2) = \begin{cases} a_{max} & \text{if } x_2 \leq 0 \\ a_{min} & \text{if } x_2 > 0 \end{cases}$$

$$\kappa = \left(\frac{1}{\sqrt{2a_{max}}} - \frac{e}{\sqrt{2a_{min}}}\right)\frac{1}{(1+e)},$$

and dom $V = \left\{x \in I\!R^2 : x_1 \geq 0\right\}$. Note that under condition (3), we have $\kappa > 0$, and that although $p(x_2)$ is piecewise-defined, $V(x)$ is still is continuous on its domain and continuously differentiable on dom $V \setminus \{\boldsymbol{O}\}$. Moreover, $V(x)$ is positive definite on $C \cup D$ since $V(x)$ is positive when $x_2 = 0$ and $x_1 > 0$, or when $x_1 = 0$ and $x_2 > 0$, and when $x_1 = 0$ and $x_2 < 0$ we have

$$V(x) \geq \sqrt{W(x)} - \kappa|x_2| \geq \frac{1}{\sqrt{2a_{max}}}|x_2| - \kappa|x_2| = \frac{e}{1+e}\left(\frac{1}{\sqrt{2a_{min}}} + \frac{1}{\sqrt{2a_{max}}}\right)|x_2| > 0\ . \tag{5}$$

Now, observe that, for all $x \in D$,

$$V(G(x)) - V(x) = \kappa(1+e)|x_2| + \frac{e}{\sqrt{2a_{min}}}|x_2| - \frac{1}{\sqrt{2a_{max}}}|x_2| = 0\ . \tag{6}$$

Moreover, for all $x \in C \setminus \{0\}$ and all $f(a) \in F(x)$, we have

$$\langle \nabla V(x), f(a)\rangle = -\kappa a + 0.5\frac{x_2}{\sqrt{W(x)}}\left(1 - \frac{a}{p(x_2)}\right) \qquad a \in [a_{min}, a_{max}]. \tag{7}$$

Since the second term in the derivatives in (7) is never positive, it follows that, for all $x \in C \setminus \{0\}$ and all $f(a) \in F(x)$, we have $\langle \nabla V(x), f(a)\rangle \leq -\kappa a_{min}$. Therefore, $V(x)$ satisfies the conditions of Proposition 1, and $\boldsymbol{O}$ is **UZAS**.

In order to prove the "and only if" part, consider the candidate Lyapunov function $W(x)$ given in (4), which is positive definite on $C \cup D$. Assume that condition (3) is violated. Therefore, for all $x \in D$, we have

$$W(G(x)) - W(x) = \left(\frac{e^2}{2a_{min}} - \frac{1}{2a_{max}}\right)x_2^2 \geq 0.$$

In the continuous part, for all $x \in C \setminus \{0\}$ and $f(a) \in F(x)$, we have

$$\langle \nabla W(x), f(a)\rangle = x_2\left(1 - \frac{a}{p_2(x_2)}\right)\ ,\ \ a \in [a_{min}, a_{max}].$$

Choosing a solution ϕ for which $a = p(x_2)$, gives $\langle \nabla W(x), f\rangle = 0$. Therefore, one can construct a solution with arbitrarily small initial condition, for which $W(x)$ is non-decreasing. Thus, this solution cannot converge to the origin. Note that in case where (3) is strictly violated, i.e. it is not satisfied as equality, W is strictly increasing on jumps, leading to *divergence* of the solution ϕ away from $\boldsymbol{O}$. ■

C. Bound on the Zeno time

We now utilize the Lyapunov analysis to derive an upper bound on the Zeno time for any given initial condition. The bound is based on the particular choice of Lyapunov function $V(x)$ given in (4), and is summarized in the following corollary.

Corollary 1: Consider all possible solutions $\phi(t,j)$ of the $\mathbf{SVBB}$ system having initial conditions $\phi(0,0) = x_0$. Then the maximal Zeno time $Z(x_0)$ is bounded by

$$Z(x_0) \leq \frac{V(x_0)}{\kappa a_{min}}, \tag{8}$$

where $V(x)$ and κ are defined in (4).

Proof: First, Eq. (6) in the proof of Theorem 1 implies that the Lyapunov function $V(x)$ does not change along the discrete jumps in any solution ϕ of (2). Moreover, (7) implies that the time-derivative of V along the continuous parts of solutions is bounded by $\dot{V} \leq -\kappa a \leq -\kappa a_{min} < 0$. Therefore, under initial condition $\phi(0,0) = x_0$, the value of V along the solution is bounded from above by $V(\phi(t,j)) \leq V(x_0) - \kappa a_{min} t$. Since $V(x)$ is positive definite and vanishes only at $\boldsymbol{O}$, the ordinary time at which V converges to $\boldsymbol{O}$ asymptotically must satisfy the bound in (8). ■

Note that (8) only provides a *conservative upper bound* on $Z(x_0)$, while its *exact* expression is derived in the next section.

IV. Optimal control analysis of Zeno stability

In this section, we utilize techniques of optimal control theory to analyze the system of the set-valued bouncing ball. First, we reproduce the proof of the condition for uniform Zeno stability in Theorem 1. Then we derive an exact expression for the maximum Zeno time $Z(x_0)$. We begin by reviewing some basic terminology and concepts of optimal control theory, and, in particular, of Pontryagin's maximum principle.

A. Review of Pontryagin's maximum principle

We now give a brief summary of Pontryagin's maximum principle. The presentation here is based on standard textbooks on optimal control theory such as [4] and [7], though we adopt here a slightly different notation. Consider a control system

$$\dot{x} = f(x,u), \tag{9}$$

where $x = (x_1, \ldots x_n) \in I\!R^n$, $u \in \Omega \subseteq I\!R^m$, and Ω is a convex set of admissible controls. A solution to (9) on a time interval $[\tau_0, \tau_f]$ is a pair $(x(t), u(t))$ satisfying (9) and $u(t) \in \Omega$ for all $t \in [\tau_0, \tau_f]$. The initial and final conditions of $x(t)$ are denoted $x_0 = x(\tau_0)$ and $x_f = x(\tau_f)$. For a given control system (9), the *optimal control problem* is defined as finding a solution to (9) under given initial conditions x_0, which maximizes a given cost function depending on the end condition $P(x_f, \tau_f)$ [1]. The solution, denoted $(x^*(t), u^*(t))$, is called the *optimal solution* associated with the given cost function. Note that the end condition x_f, as well as the end time τ_f, may be either specified or left as free parameters of optimization.

The optimal control problem can be formulated as a problem in calculus of variations, and its solution is based on the classical notion of Pontryagin's maximum principle, which is stated as follows. First, define the *co-state vector* $\lambda(t) \in I\!R^n$. Next, define the system's *Hamiltonian*, given by $H(x, u, \lambda, t) = \lambda(t)^T f(x, u)$. The co-state dynamic equation is then given by

$$\dot{\lambda} = -\frac{\partial H}{\partial x}. \tag{10}$$

The optimal control input $u^*(t)$ is given by

$$u^*(t) = \text{argmax}_u \, H(x(t), u(t), \lambda(t), t) \tag{11}$$

for $t \in [\tau_0, \tau_f]$. The optimal solution is obtained by solving the coupled equations (9), (10), and (11) under boundary conditions $x(\tau_0) = x_0$ and $x(\tau_f) = x_f$. In case where the end condition for $x_{i,f} = x_i(\tau_f)$ is not specified for some $i \in \{1 \ldots n\}$, an alternative end condition for λ_i is given by

$$\lambda_i(\tau_f) = \frac{\partial P}{\partial x_{i,f}}. \tag{12}$$

Finally, in case where the end time τ_f is also not specified, an additional condition on $H(\tau_f)$ is given by

$$H(x(\tau_f), u(\tau_f), \lambda(\tau_f), \tau_f) = -\frac{\partial P}{\partial \tau_f}. \tag{13}$$

B. Formulating the $\mathbf{SVBB}$ *as an optimal control problem*

We now formulate the continuous part of the dynamics of the $\mathbf{SVBB}$ as an optimal control problem, as follows. The control system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \quad \text{where } u \in [-a_{max}, -a_{min}], \tag{14}$$

represents the set-valued differential equation $\dot{x} \in F(x)$, where $F(x)$ is given in (2). We view the initial and final times τ_0 and τ_f as the endpoints of a time interval $[t_j, t_{j+1}]$ in a solution $\phi(t, j)$ of the $\mathbf{SVBB}$ system. The initial condition of (14) is thus given by $x(\tau_0) = (0, v)$ for some $v \in I\!R_{\geq 0}$. One end condition is specified, namely $x_1(\tau_f) = 0$. However, the end time τ_f, as well as $x_2(\tau_f)$, which corresponds to the ball's terminal velocity, are both unspecified. The Hamiltonian of this system is given by

$$H(x, \lambda, u, t) = \lambda_1 x_2 + \lambda_2 u. \tag{15}$$

Using (10), the co-state dynamic equations are then given by

$$\dot{\lambda}_1 = 0, \;\; \dot{\lambda}_2 = -\lambda_1, \tag{16}$$

[1]Many textbooks also consider an integral cost function of the form $J = \int_{\tau_0}^{\tau_f} g(x, u, t)dt$. This cost function can be easily incorporated into the formulation here by augmenting the state vector x with an additional variable z whose dynamics is given by $\dot{z} = g(x, u, t)$. The cost function is then simply given by $P = z(\tau_f)$.

indicating that $\lambda_1(t)$ is constant and $\lambda_2(t)$ is a linear function. Pontryagin's maximum principle (11) then implies that the optimal control input $u^*(t)$ can only be one of the endpoints $-a_{min}$ or $-a_{max}$ (a case known as "bang-bang" control, which is typical for linear systems under optimal control). Moreover, the choice of $u^*(t)$ is determined only by the sign of $\lambda_2(t)$, where $u^*(t) = -a_{max}$ if $\lambda_2(t) < 0$, and $u^*(t) = -a_{min}$ if $\lambda_2(t) > 0$. Since $\lambda_2(t)$ is a linear function, it has at most one zero-crossing point in the time interval $[\tau_0, \tau_f]$. Therefore, the optimal control $u^*(t)$ is piecewise-constant, with *at most one switching time*. The proofs of all results in the rest of this section will build on this setup for deriving optimal solutions under different choices of a cost function.

C. Proof of Zeno stability condition via optimal control analysis

We now reproduce the proof of Theorem 1 which gives the condition for $\mathbf{UZAS}$ of the $\mathbf{SVBB}$, by using optimal control analysis.

Proof [of Theorem 1]: The key idea of the proof is to consider the sequence of velocities x_2 at the discrete times t_j in all possible solutions $\phi(t, j)$, find the "most unstable" possible sequence, and require that it decays asymptotically to zero as $j \to \infty$. For a given solution $\phi(t, j) = (x_1(t, j), x_2(t, j))$ of the $\mathbf{SVBB}$ system under initial condition $\phi(0, 0) = x_0$, denote $v_j = x_2(t_j, j)$ for $j \in \{1, 2 \ldots\}$. Physically, v_j is the post-impact velocity of the ball right after the jth collision with the ground. By construction, one has $v_j \geq 0$. First, we show that for any possible solution $\phi(t, j)$, the sequence of v_j satisfies

$$\frac{e}{\sqrt{\alpha}} v_j < v_{j+1} < e\sqrt{\alpha} v_j, \tag{17}$$

where α is defined in (3). In order to prove (17), we consider the optimal control system (14), which describes the evolution of $\phi(t, j)$ in a specific time interval $[t_j, t_{j+1}]$. The initial condition is given by $x(\tau_0) = (0, v)$, and the end condition is $x_1(\tau_f) = 0$, where the end velocity $x_2(\tau_f) = x_{2f}$ is unspecified. We seek for the solution $(x^*(t), u^*(t))$ that maximizes the absolute value of x_{2f}. Since $x_{2f} < 0$, the cost function to be maximized is chosen as $P(x_f, \tau_f) = -x_{2f}$. Since x_{2f} is unspecified, the end condition on λ_2 in (12) gives $\lambda_2(\tau_f) = -1$, and the maximum principle (11) then implies that $u^*(\tau_f) = -a_{max}$. Since we concluded that $u^*(t)$ switches between a_{max} and a_{min} at no more than a single point, one can write $u^*(t) = -a_{min}$ for $t \in [\tau_0, \tau_s)$ and $u^*(t) = -a_{min}$ for $t \in [\tau_s, \tau_f]$, where τ_s is an unknown switching time. By substitution of $u = u^*(t)$ and direct integration of (14) under the given initial and end conditions, one can solve for the end time τ_f by substituting $x_1(\tau_f) = 0$. The solution for the terminal velocity x_{2f} is then obtained as

$$x_{2f} = -\sqrt{(v - a_{min}\tau_s)^2 + 2a_{max}(v\tau_s - a_{min}\tau_s^2/2)}. \tag{18}$$

Using elementary calculus, it is then straightforward to show that x_{2f} attains a minimal value of $x_2^*(\tau_f) = -\sqrt{\frac{a_{max}}{a_{min}}} v$, under the critical switching time $\tau_s^* = v/a_{max}$. One can also verify that $x_2^*(\tau_s^*) = 0$. The physical meaning of this optimal solution is selecting the "slowest" acceleration $\dot{x}_2 = -a_{min}$ when the ball is on its way up, i.e. $x_2 > 0$, and the "fastest" acceleration $\dot{x}_2(t) = -a_{max}$ for the way down, i.e. $x_2 > 0$. Under these selections, one attains maximal hitting velocity at the ground collision. Interestingly, the optimal input is precisely $u^* = -p_2(x_2)$, where $p_2(x_2)$, defined in (4), was used in the derivation of the Lyapunov function in the previous section. Finally, setting $v = v_j$, applying the jump rule $v_{j+1} = -ex_2(t_{j+1}, j) = -ex_2(\tau_f)$ and using the definition for α in (3) completes the proof of the upper bound in (17). The proof of the lower bound in (17) is obtained in a similar way, by maximizing x_{2f}. As a result, for any solution $\phi(t, j)$, the sequence v_j is bounded between the two geometric series given by

$$v_1 \left(\frac{e}{\sqrt{\alpha}}\right)^{j-1} \leq v_j \leq v_1 \left(e\sqrt{\alpha}\right)^{j-1}. \tag{19}$$

In order to get asymptotic convergence of $\phi(t, j)$ to $\boldsymbol{O}$, the sequence v_j must decay to zero. This is satisfied when the factor of the upper-bounding geometric series in (19) is less than one, which is precisely condition (3) in Theorem 1. Note that this condition is *necessary and sufficient*, as we have proved that the upper bound in (17) is tight, since the particular solution $\phi(t, j)$ with $a = p(x_2)$ satisfies the upper bound in (19) as an equality.

Next, we show that all solutions $\phi(t, j)$ have a finite bound on ordinary time $T(\phi) < \infty$. Consider again the optimal control problem (14), under initial condition $x(\tau_0) = (0, v)$. The cost function to be maximized is chosen as $P(x_f, \tau_f) = \tau_f$. The end condition on λ_2 in (12) gives $\lambda_2(\tau_f) = 0$. The end condition (13) gives $H(\tau_f) = -1$. Using the expression for H in (15) and the fact that $x_{2f} < 0$, we conclude that $\lambda_1(\tau_f) > 0$. Using the solution for $\lambda_i(t)$ in (16), we conclude that $\lambda_2(t) > 0$ for all $t \in ([\tau_0, \tau_f)$. The maximum principle (11) then implies that the optimal input is constantly $u^* = -a_{min}$, without any switches. It can then be shown that the maximal time is given by $\tau_f^* = 2v/a_{min}$. Applying this result to a time interval $[t_j, t_{j+1}]$ in a solution $\phi(t, j)$ of the $\mathbf{SVBB}$, the time difference $\Delta_j = t_{j+1} - t_j$ is bounded by $\Delta_j \leq 2v_j/a_{max}$. Using the upper bound for v_j in (19), one obtains an upper bound on the Zeno time under initial condition $x_0 = (h_0, v_0)$ as

$$\begin{aligned} &T(\phi) = t_1 + \textstyle\sum_{j=1}^{\infty} \Delta_j \leq t_1 + \sum_{j=1}^{\infty} 2\dfrac{v_j}{a_{min}} \leq t_1^* + 2\dfrac{v_1^*}{a_{min}(1 - e\sqrt{\alpha})}, \\ &\text{where } t_1^* = \frac{v_0 + \sqrt{v_0^2 + 2a_{min}x_{10}}}{a_{min}}, \text{ and } v_1^* = \begin{cases} e\sqrt{\alpha(v_0^2 + 2a_{min}h_0)} & v_0 \geq 0 \\ e\sqrt{v_0^2 + 2a_{max}h_0} & v_0 < 0 \end{cases} \end{aligned} \tag{20}$$

Therefore, under the condition (3), all possible solutions are Zeno and converge to $\boldsymbol{O}$ in finite time, which, by varying the initial condition, can be made arbitrarily small. ∎

Note that the bound on the Zeno time in (20) is not tight, just as the one given in (8). The reason for that is the fact that the maximum time in (20) is obtained by selecting the constant input $u^* = -a_{min}$, while the "most unstable" velocity sequence v_j is obtained by taking $u^* = -p(x_2)$. Therefore, a solution $\phi(t, j)$ such that $T(\phi)$ actually attains the bound in (20) does not necessarily exist.

D. Tight bound on Zeno time of the $\mathbf{SVBB}$

We now present the second main contribution of this work, which is an exact tight bound on the Zeno time of solutions of the $\mathbf{SVBB}$ system under any given initial condition. As a first step, we focus on initial conditions of the form $\phi(0, 0) = (0, \nu)$ for $\nu > 0$, corresponding to the ball starting initially on the ground with nonzero upward velocity. The bound on Zeno times under initial conditions of this form is summarized in the following lemma.

Lemma 1: *Consider all possible solutions $\phi(t,j)$ of the* $\mathbf{SVBB}$ *system under initial condition $\phi(0,0)=x_0=(0,\nu)$, where $\nu>0$. Assuming that condition (3) is satisfied, all solutions are Zeno, and their maximal Zeno time $Z(x_0)$ is given by*

$$Z(x_0)=2\frac{1+e}{1-e^2\alpha}\cdot\frac{\nu}{a_{min}}. \tag{21}$$

Moreover, there exists a particular solution $\phi^*(t,j)$ such that $T(\phi^*)=Z(x_0)$.

Proof: Recall that under condition (3), Theorem 1 implies that all possible solutions $\phi(t,j)$ are Zeno. We therefore seek for an *optimal* solution ϕ that maximizes $T(\phi)$ for given ν. Let $\phi^*_\nu(t,j)$ denote this solution. We now make two key observations, as follows. First, we note that the $\mathbf{SVBB}$ system satisfies the property of *homogeneity* [14]. In particular, the linearity and time-invariance (LTI) of both discrete-time and continuous-time components of (2) (the continuous-time part is represented in (14) as a control system which is also LTI), as well as the homogeneity of the sets C and D, imply that for any $c>0$, $\phi(t,j)$ is a solution of (2) if and only if $c\phi(t/c,j)$ is a solution of (2). (This is a special case of Lemma 3.4 in [14], see also [25].) Therefore, for any $c>0$ one obtains that $\phi^*_{c\nu}(t,j)=c\phi^*_\nu(t/c,j)$.

A second observation is that any "tail" of an optimal solution is also an optimal solution. Therefore, denoting $\phi^*_\nu(t_k,k)=(0,v_k)$ for some $k\in I\!N$, one obtains $\phi^*_{v_k}(t,j)=\phi^*_\nu(t+t_k,j+k)$. These two observations together imply the existence of a scalar $\eta\in(0,1)$ such that for any $(t,j)\in\text{dom }\phi^*_\nu$, one has

$$\phi^*_\nu(t,j)=\eta^j\phi^*_\nu(\eta^{-j}(t-t_j),0). \tag{22}$$

That is, the behavior of the optimal solution in the jth interval of ordinary time is identical to its behavior in the first time interval up to scaling of the magnitude and time. Therefore, the problem of finding the optimal solution $\phi^*_\nu(t,j)$ reduces to solving an optimal control problem on the first time interval $[0,t_1]$ only. The scalar η is then given by $\eta=v_1/\nu=-ex^*_2(t_1)/\nu$, and the remainder of the optimal solution is simply obtained by using (22). In particular, since the discrete times t_j in the optimal solution $\phi^*_\nu(t,j)$ satisfy the geometric series relation $t_{j+1}-t_j=\eta^j t_1$, the Zeno time is given by the sum

$$T(\phi^*)=\frac{t_1}{1-\eta}. \tag{23}$$

Consider again the control system (14), which represents the solution $\phi^*_\nu(t,j)$ in the time interval $[0,t_1]$. We now solve an optimal control problem, where the cost function to be maximized is the Zeno time (23), which is formulated here as $P(x_f,\tau_f)=\frac{\tau_f}{1+ex_{2f}/\nu}$. The solution of the co-state dynamics $\lambda_i(t)$ is given in (16). Using (12), the end condition for λ_2 is given by $\lambda_2(\tau_f)=-\frac{e\tau_f\nu}{(\nu+ex_{2f})^2}<0$. Therefore, the maximum principle (11) implies that $u^*(\tau_f)=-a_{max}$. Thus, the optimal input satisfies $u^*(t)=-a_{min}$ for $t\in[\tau_0,\tau_s)$ and $u^*(t)=-a_{min}$ for $t\in[\tau_s,\tau_f]$, where τ_s is an unknown switching time. By substitution of $u=u^*(t)$ and direct integration of (14) under the given initial and end conditions, one can solve for the end time τ_f and obtain the solution for the terminal velocity x_{2f} as

$$\begin{aligned} x_{2f}&=-\sqrt{(\nu-a_{min}\tau_s)^2+2a_{max}(\nu\tau_s-a_{min}\tau_s^2/2)}\\ \tau_f&=\tau_s+\frac{\nu-a_{min}\tau_s+\sqrt{(\nu-a_{min}\tau_s)^2+2a_{max}(\nu\tau_s-a_{min}\tau_s^2/2)}}{a_{max}}. \end{aligned} \tag{24}$$

Using (24), the cost function P can now be expressed in terms of the variable switching time τ_s. Using elementary calculus, it is then straightforward to show that $P(\tau_s)$ attains a maximal value for $\tau_s^* = 2\frac{\nu}{a_{min}} \cdot \frac{1+e}{1+2e+e^2\alpha}$. Substitution of $\tau_s = \tau_s^*$ into the expression for P then gives the maximal Zeno time in (21). ■

We now use Lemma 1 to establish the main result of this section, which is an exact tight bound on the Zeno time of the $\mathbf{SVBB}$ system under *any* initial condition. The result is summarized in the following theorem.

Theorem 2: *Consider all possible solutions $\phi(t,j)$ of the $\mathbf{SVBB}$ system under initial condition $\phi(0,0) = x_0 = (h_0, v_0)$. Assuming that condition (3) is satisfied, all solutions are Zeno, and their maximal Zeno time $Z(x_0)$ is given by*

$$Z(x_0) = \begin{cases} \dfrac{v_0 + \sigma U_{0min}}{a_{min}} & v_0 \geq v_c \\ \dfrac{v_0 + U_{0max}(1+\beta\alpha)}{a_{max}} & v_0 < v_c \end{cases} \tag{25}$$

where

$$U_{0max} = \sqrt{v_0^2 + 2a_{max}h_0}\ ,\ U_{0min} = \sqrt{v_0^2 + 2a_{min}h_0}\ ,\ v_c = -\sqrt{\frac{2a_{min}h_0}{\sigma^2 - 1}}$$

$$\alpha = \frac{a_{max}}{a_{min}}\ ,\ \beta = 2e\frac{1+e}{1-e^2\alpha}\ ,\ \sigma = \sqrt{1 + 2\beta + \alpha\beta^2}. \tag{26}$$

Moreover, there exists a particular solution $\phi^*(t,j)$ such that $T(\phi^*) = Z(x_0)$.

Proof: We are seeking for the solution $\phi^*(t,j)$ that maximizes the Zeno time $T(\phi)$ under initial condition $x_0 = (h_0, v_0)$. Consider the "tail" of the solution $\phi^*(t,j)$ for $t \geq t_1$, which has initial condition $(0, v_1)$. Using Lemma 1, the maximal Zeno time of this solution, denoted T_1^*, is obtained by substituting $\nu = v_1$ in (21). Therefore, one only needs to consider the first time interval $[0, t_1]$ of $\phi^*(t,j)$ as an optimal control problem of the system (14) where $\tau_f = t_1$, and optimize the cost function $t_1 + T_1^*$. The velocity v_1 is related to the end condition of the control system (14) via the relation $v_1 = -ex_{2f}$. Therefore, using the definition of β in (26), we choose the cost function to be maximized as

$$P(x_f, \tau_f) = \tau_f - \beta\frac{x_{2f}}{a_{min}}, \tag{27}$$

which is precisely $t_1 + T_1^*$. The Hamiltonian of the system is defined in (15), and the dynamics of the co-state variables λ_1, λ_2 is given in (16). The end condition for $\lambda_2(t)$ in (12) gives $\lambda_2(t_f) = -\frac{\beta}{a_{min}} < 0$. The maximum principle (11) then implies that $u^*(\tau_f) = -a_{max}$. Thus, the optimal input is taken as $u^*(t) = -a_{min}$ for $t \in [\tau_0, \tau_s)$ and $u^*(t) = -a_{min}$ for $t \in [\tau_s, \tau_f]$, where τ_s is an unknown switching time. Note that the case $\tau_s = 0$ corresponds to taking constant input $u^* = -a_{max}$ without switching. Substituting $u = u^*(t)$ into (14), direct integration gives

$$\tau_f = \tau_s + \frac{v_s + \sqrt{v_s^2 + 2a_{max}h_s}}{a_{max}},\ x_2(\tau_f) = v_s - (\tau_f - \tau_s)a_{max},$$

where $v_s = v_0 - a_{min}\tau_s$ and $h_s = h_0 + v_0\tau_s - \frac{1}{2}a_{min}\tau_s^2$.

Substituting these expressions, the cost function $P(x_f, t_f)$ is now a function of τ_s only. Using elementary calculus, it is then straightforward to show that $P(\tau_s)$ attains a maximum at the critical switching time given by

$$\tau_s^* = \frac{\sigma v_0 + U_{0min}}{\sigma a_{min}},$$

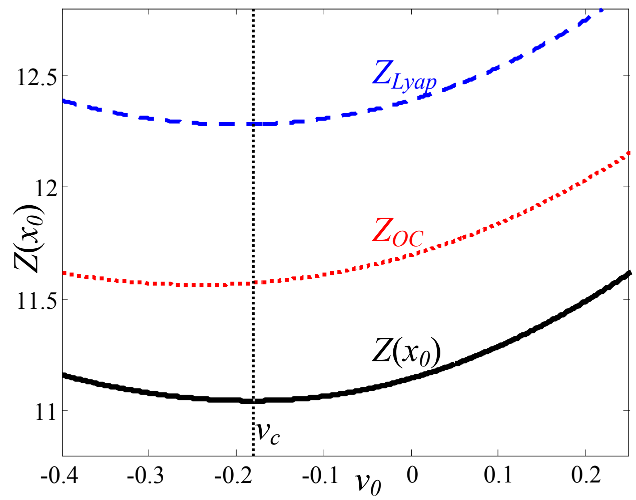

Fig. 1. Plot of the exact bound on Zeno time $Z(x_0)$ (solid) as a function of , compared to the estimates Z_{Lyap} in (8) (dashed) and Z_{OC} in (20). Parameter values are $e = 0.7$, $a_{min} = 0.9$, $a_{max} = 1.1$ and $h_0 = 1$.

where U_{0min} and σ are defined in (26). Note that in order to get a nonnegative switching time $\tau_s^* \geq 0$, v_0 must satisfy $\sigma v_0 + U_{0min} \geq 0$. In case where $\sigma v_0 + U_{0min} < 0$, one gets $\tau_s^* < 0$, a the optimal solution is obtained by taking $\tau_s = 0$, so that $u^*(t) = -a_{max}$ at all times, without switching. Finally, direct substitution of the optimal solution into the cost function (27) gives the piecewise-defined expression in (25), where the condition $\sigma v_0 + U_{0min} \geq 0$ is reformulated as $v_0 \geq v_c$. ■

Example: We now demonstrate the result by computing the exact bound on the Zeno time and compare it with its estimates in (8) and (20). The parameters of the system's data in (2) are chosen as $e = 0.7$, $a_{min} = 0.9$, and $a_{max} = 1.1$. We compute the exact bound $Z(x_0)$ according to (25) for initial conditions $x_0 = (h_0, v_0)$, where we fix $h_0 = 1$ and vary v_0. Fig. 1 shows $Z(x_0)$ as a function of v_0, appearing as a solid curve. The value of v_c in (26) for this data is $v_c = -0.181$. The dashed curve in the plot is the estimate Z_{Lyap} derived in (8) via Lyapunov analysis, and the dotted curve is the estimate Z_{OC}, derived in (20) using optimal control considerations. It can be seen that the estimates Z_{Lyap} and Z_{OC} are both conservative, with deviations of approximately 10% and 5%, respectively, from the exact bound $Z(x_0)$.

V. Conclusion

In this work we studied the hybrid dynamics of the set-valued bouncing ball. We used Lyapunov analysis and optimal control techniques to obtain a necessary and sufficient condition for Zeno stability, and derived an exact bound on the maximal Zeno time as for a given initial conditions. The result is useful for obtaining bounds on Zeno solutions of nonlinear single-valued hybrid systems in the vicinity of Zeno equilibria. Two future directions for extension of the results are as follows. First, consideration of set-valued dynamics that are more complicated than the linear dynamics of the **SVBB** may enable a closer approximation of true nonlinear hybrid systems, in order to obtain tighter bounds on their Zeno solutions. Second, utilization of the results for planning cyclic tasks that involve Zeno behavior in robotic systems, in the spirit of [22], as well as stabilization and control of such tasks, are challenging open problems.

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